Math 208
Potential Functions/Gradient Fields/Path Independence Problem set

In problems 1–2, show that $\mathbf{F}$ is a conservative vector field (for each simply connected region in its domain), and find a potential function f for $\mathbf{F}$.

1. $\mathbf{F}(x,y) = (2\frac{y}{x} - 5x^4 + 3)\mathbf{i} + (2\ln(x) + \sin(y))\mathbf{j}$

2. $\mathbf{F}(x,y,z) = (2x - y - 2z)\mathbf{i} + (y - x + 4z)\mathbf{j} + (4y - 2x + 7)\mathbf{k}$

In problems 3–4, show that $\mathbf{F}$ is the gradient of some function f, and then find such an f.

3. $\mathbf{F}(x,y,z) = (6\frac{xy}{z} - 5)\mathbf{i} + (3\frac{x^2}{z} - 2z)\mathbf{j} - (3\frac{x^2 y}{z^2} + 2y + e^{2z})\mathbf{k}$

4. $\mathbf{F}(x,y) = (x - 5)\mathbf{i} + (3y^2 + 7)\mathbf{j}$

In problems 5–6, show that the differential form is exact and find the corresponding potential function.

5. $(x\cos(2y) - 4xz)dx - (x^2\sin(2y) + 5)dy + (21z^2 - 3 - 2x^2)dz$

6. $(\frac{yz^2}{x^2+1} + 3z + 2)dx + (z^2\tan^{-1}(x) + z^{-1} - 1)dy + (2yz\tan^{-1}(x) + 3x - yz^{-2} + 3)dz$

In problems 7-9, show that the integral is independent of path, and then find its value using a potential function.

7. $\int_{(-1,3)}^{(2,1)} ((x^2 y - 3)\,dx + (\frac{1}{3}x^3 - 4)\,dy)$

8. $\int_{(1,-2,3)}^{(4,2,0)} ((5z + 4)\,dx - (2y^3 - z)\,dy + (5x + y + 4)\,dz)$

9. $\int_{(2,-1,-2)}^{(1,2,1)} (y^2 z^3 dx + (2xyz^3 - 4y)\,dy + (3xy^2 z^2 - 2)\,dz)$

Answers:

1. $2y\ln(x) - x^5 + 3x - \cos(y)$

2. $x^2 - xy - 2xz + \frac{y^2}{2} + 4yz + 7z$

3. $3x^2yz^{-1} - 5x - 2yz - \frac{e^{2z}}{2}$

4. $\frac{x^2}{2} - 5x + y^3 + 7y$

5. $\frac{x^2}{2}\cos(2y) - 2x^2z - 5y + 7z^3 - 3z$

6. $yz^2\tan^{-1}(x) + 3xz + 2x + yz^{-1} - y + 3z$

7. $8/3$. Note the potential function is $f(x,y) = \frac{1}{3}x^3y - 3x - 4y$.

8. -9. Note the potential function is $f(x,y,z) = 5xz + 4x - \frac{1}{2}y^4 + yz + 4z$.

9. 8. Note the potential function is $f(x,y,z) = xy^2z^3 - 2y^2 - 2z$.