Supplementary exercises

These are some supplementary exercises for use with the revised ALAMA textbook.

Chapter 1:

Section 1.1

Exercise 1. View the network Problem 18 of this section as a web of five pages with links indicated by arrows.

(a) According to the first pass at page ranking, what is the ranking of these pages?

(b) Set up the system of equations that results from the second pass at page ranking of this web.

(c) Set up the system of equations that results from the third pass at page ranking of this web.

Section 1.2

Problem 1. The complex exponential function of the variable $z = x + y\mathrm{i}$ is defined by the formula $e^z = e^x e^{\mathrm{i}y}$. An increasingly better approximation to this function is given by $e^z \approx \sum_{k=0}^{n} \frac{z^k}{k!}$, with convergence as $n \to \infty$. Assume this and derive approximations for $\sin\theta$ and $\cos\theta$ from this formula.

Section 1.3

Exercise 1. Use Gauss-Jordan elimination to solve the system resulting from the second and third approach to page ranking as applied to Exercise 1 of Section 1.1 in these supplementary exercises. Discuss your results.

Section 1.4

Exercise 1. Find the RREF, rank and nullity of the following matrix:

$$A = \begin{bmatrix} 3 & 1 & -2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & -1 & 1 & 2 & 2 \\ 3 & 2 & -1 & 1 & 1 & 8 & 9 \\ 0 & 2 & 2 & -1 & 1 & 6 & 8 \\ 0 & 3 & 3 & 3 & -3 & 0 & 3 \end{bmatrix}.$$

Section 1.5

Problem 1. If we start with a fixed amount of material at stage $k = 0$ with zero source term and use Equation (1.11) of the textbook on p. 58 to solve for $y_{i,j+1}$, $i = 1, \ldots, n$, and assume that $y_{0,j} = 0 = y_{n+1,j}$ at every step j, then it can be argued that this method is losing material at each step. Confirm this by adding up the total amount of material at the jth and $(j+1)$th stages.

Chapter 2:

Section 2.1

Section 2.2

Exercise 1. Fill in the blanks or show it is not possible:

$$\begin{bmatrix} 1 \\ 2 \\ 9 \\ 8 \\ 3 \end{bmatrix} = \underline{\quad\quad} \begin{bmatrix} -2 \\ 0 \\ -1 \\ 2 \\ 3 \end{bmatrix} + \underline{\quad\quad} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 3 \end{bmatrix} + \underline{\quad\quad} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

Section 2.3

Section 2.4

Section 2.5

Exercise 1. Find A^{-1} where

(a)$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 2 & 5 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 \end{bmatrix}$

Section 2.6

Exercise 1. Find all values of $c$and d such that the matrix A invertible, where

$$A = \begin{bmatrix} c & 1 \\ 0 & 1+\mathrm{i} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & d\mathrm{i} & 1 \end{bmatrix}.$$

Exercise 2. Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 6 & 0 \\ 5 & -3 & 0 & 5 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 4 & 3 & 2 & 4 & 7 \\ 0 & -1 & 0 & 7 & 0 \end{bmatrix}$.

Section 2.7

Section 2.8

Chapter 3:

Section 3.1

Section 3.2

Exercise 1. Determine which of the following subsets V of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$ and give reasons for your answers:

(a) $V = \{(s,t,t^2) \mid s,t \in \mathbb{R}\}$

(b) $V = \{\mathbf{x} \in \mathbb{R}^3 \mid A\mathbf{x} = \mathbf{0}\}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 8 & 6 & 2 \end{bmatrix}$.

(c) $V = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x}^T A\mathbf{x} = \mathbf{0}\}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Exercise 2. Let $V = \mathbb{R}^{2,2}$ be the space of all real 2×2 matrices. Is the subset of noninvertible matrices a subspace of V? Justify your answer.

Section 3.3

Exercise 1. Determine if the following sets S of vectors are linearly independent and give reasons for your answers:

(a) $V = \mathbb{R}^3$, $S = \{(1,1,2), (1,2,3), (2,2,4)\}$

(b) $V = \mathbb{R}^4$, $S = \{(1,2,3,4), (4,2,3,1), (9,0,6,2), (4,2,7,8), (1,5,9,1)\}$

(c) $V = C[0,1]$, $S = \{x^2+1, x, \sin x, x^3+x^2\}$

(d) $V = \mathcal{P}_2$, $S = \{1-2x+2x^2, 3+x, 5-x+4x^2\}$

Section 3.4

Exercise 1. Let $T : V \to W$ be a linear operator from the vector space V to vector space W. If $\dim V = 9$ and $\dim W = 4$, what are the possible values for the dimension of $\ker(T)$. Justify your answer.

Exercise 2. Let $T : V \to \mathbb{R}^3$ be a linear operator from the vector space V with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to vector space $\mathbb{R}^3$ such that $T(\mathbf{v}_1) = (1,1,0)$, $T(\mathbf{v}_2) = (0,1,1)$ and $T(\mathbf{v}_3) = (1,0,-1)$. Answer the following and justify your answer:

(a) Is either of the vectors $(5,2,-3)$ or $(7,-5,2)$ in the range $\mathcal{R}(T)$?

(b) Is either of the vectors $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ or $\mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3$ in the kernel $\ker(T)$ of T?

Section 3.5

Exercise 1. Exhibit a subspace V of $\mathbb{R}^6$ such that $(1,0,1,0,1,0) \in V$ and $\dim V = 3$. Justify your answer.

Exercise 2. Find a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,2,1)$, $(0,1,0)$, $(1,0,1)$, $(2,1,2)$.

Exercise 3. Let V be a vector space of dimension n and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k\}$ be a subset of V. Answer True/False.

(a) If S is a basis of V, then $k = n$.

(b) if S spans V, then $k \le n$.

(c) If S is linearly independent, then $k \le n$.

(d) If S is linearly independent and $k = n$, then S spans V.

(e) If S spans V and $k = n$, then S is a basis for V.

Section 3.6

Exercise 1. Suppose that R is the RREF of A, where

$$A = \begin{bmatrix} 3 & 3 & 1 & 6 & 0 & 13 & 1 \\ 2 & 2 & -3 & 4 & 0 & -6 & 2 \\ -1 & -1 & -1 & -2 & 0 & -7 & -2 \\ 1 & 1 & 3 & 2 & 0 & 15 & -1 \\ -1 & -1 & -3 & -2 & 2 & -13 & -3 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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(a) Find a basis for $\mathcal{R}(A)$.

(b) Find a basis for $\mathcal{C}(A)$.

(c) Find a basis for $\mathcal{N}(A)$.

Section 3.7

Section 3.8

Exercise 1. Consider the problem of minimizing $C = 2x_1 + 3x_2$ subject to the constraints $x_i \ge 0$, $i = 1,2$, and $x_1 + 2x_2 \ge 4$, $x_1 + x_2 \ge 3$.

(a) Solve this problem by the graphical method.

(b) Convert this problem to a maximization problem for $-C$ and solve it by the simplex method.

(c) Use duality to convert this problem to a maximization problem and solve it.

Exercise 2. Consider the following problem:

Maximize $P = 4x_1 + 2x_2 + 3x_3 + 5x_4$ subject to constraints

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 2x_2 &= 300 \\ 8x_1 + x_2 + x_3 + 5x_4 &= 300 \\ x_j &\ge 0, \ \ j = 1,2,3,4. \end{aligned}$$

(a) Solve this problem using artificial variables and the last strategy described in Example 3.56 of the textbook.

(b) Use the constraints to eliminate two variables and solve the resulting two variable problem by the simplex method.

(c) Solve the two variable problem of (b) by the graphical method.

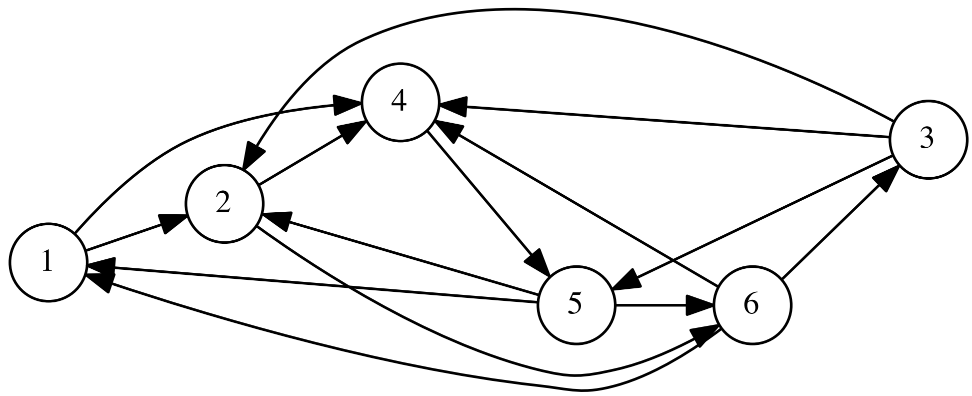

FIGURE 1. Directed Graph for Exercise 3.9.1

Section 3.9

Problem 1. Find all algebraic and directed loops of Figure 1.

Chapter 4:

Section 4.1

Section 4.2

Exercise 1. Find the projection of $\mathbf{u} = (2, 0, 4, 1)$ along the vector $\mathbf{v} = (1, 1, 1, 1)$ and use this to express $\mathbf{u}$as a sum of a vector parallel to $\mathbf{u}$ and a vector orthogonal to $\mathbf{u}$. Calculate the lengths of all vectors in this problem.

Section 4.3

Exercise 1. Apply the Gram-Schmidt algorithm to the following vectors in $\mathbb{R}^4$ to obtain an orthogonal set of vectors and use this to obtain an orthonormal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$:

$$\{(1, 1, 1, 1), (4, 2, 4, 2), (0, 0, 0, 2)\}.$$

Problem 2. Show that if P and Q are orthogonal matrices of the same size, then PQ is also orthogonal.

Section 4.4

Chapter 5:

Section 5.1

Exercise 1. You are given that $\lambda = 1$ and $\lambda = 3$ are eigenvalues for the matrix $\begin{bmatrix} -7 & 1 & 4 \\ 0 & 1 & 0 \\ -20 & 2 & 11 \end{bmatrix}$. Use this to find an eigensystem for A.

Problem 2. Use a technology tool to compute the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Based of the results, what is the multiplicity of the eigenvalue 0? Can you confirm your answer conclusively?

Section 5.2

Exercise 1. Find the characteristic polynomial, a matrix P and diagonal matrix D such that $P^{-1}AP = D$ for A, where

(a) $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Problem 2. Prove that if A is a diagonalizable matrix, then so is A^T.

Section 5.3

Exercise 1. Find an eigensystem for $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -4 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and use it to compute A^{98}.

Section 5.4

Exercise 1. Find an orthogonal matrix P that diagonalizes $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and use this to find a formula for A^k, k a positive integer.

Section 5.5

Problem 1. The Schur theorem generalizes to apply to non-square $m \times n$ matrices A with $m < n$ as follows: Given such a matrix there exists a unitary $n \times n$ matrix U such that V^*AU is upper triangular with V equal to the $m \times m$ submatrix of the upper left corner of U. Devise a strategy that confirms this fact and apply it to the 4×6 matrix A whose (i, j)th entry is $i + j - 1$ by using a technology tool.

Section 5.6

Section 5.7

Problem 1. Show how the algebraic and geometric multiplicities of an $n \times n$ matrix A can be determined by inspection of its Jordan canonical form.

Problem 2. The *minimal polynomial* of an $n \times n$ matrix A is defined to be the monic polynomial $p(x)$ of least degree such that $p(A) = \mathbf{0}$.

(a) Give an example of a matrix A such that the degree of the characteristic polynomial of A is larger than the degree of the minimal polynomial of A.

(b) Determine the characteristic and minimal polynomials of matrix A in terms of its Jordan canonical form and show that the minimal polynomial is a divisor of the characteristic polynomial.

Chapter 6:

Section 6.1

Problem 1. Show that the ball $B_r(\mathbf{v}_0)$ in the normed linear space V is *convex*, i.e., for vectors $\mathbf{u}, \mathbf{v} \in B_r(\mathbf{v}_0)$, the convex combination $\mathbf{w} = \lambda \mathbf{u} + (1-\lambda)\mathbf{v}$, where $0 \le \lambda \le 1$, belongs to $B_r(\mathbf{v}_0)$.

Section 6.2

Exercise 1. For a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in an inner product space V, define the $n \times n$ matrix H by $H = [\langle \mathbf{v}_i, \mathbf{v}_j \rangle]$. Let $V = C[0,1]$ with the standard inner product and define $\mathbf{v}_i = x^{i-1}$, $i = 1, 2, \ldots, n$. Compute the matrix H for this set of vectors and show that the resulting matrix is the Hilbert matrix.

Exercise 2. Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and find the angle between vectors $(1,0)$ and $(0,1)$ in the inner product space $V = \mathbb{R}^2$ with inner product $\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u})^T (A\mathbf{v})$.

Problem 3. With H as in Exercise 1, show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in an inner product space V are linearly independent if and only if H is nonsingular. Use this to show that the Hilbert matrix has nonzero determinant.

Section 6.3

Exercise 1. Let $V = \mathbb{R}^2$ with inner product defined by $\langle (u_1, u_2), (v_1, v_2) \rangle = 2u_1v_1 + 3u_2v_2$. Let $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (2, 1)$ and $W = \operatorname{span}\{\mathbf{u}\}$. Compute the following:

(a) $\|\mathbf{u}\| =$

(b) $\langle \mathbf{u}, \mathbf{v} \rangle =$

(c) $\operatorname{proj}_W \mathbf{v} =$

Section 6.4

Section 6.5

Problem 1. Show that for a matrix A, $\|A\|_2 = \sigma_1(A)$, i.e., the largest singular value of A.

Problem 2. Show that for an $m \times n$ matrix A, $\min_{\|\mathbf{x}\|_2 = 1} \|A\mathbf{x}\|_2 = \sigma_n(A)$, i.e., the smallest singular value of A.

Section 6.6

Problem 1. Example 6.22 contains an infinite series formula for calculationg π^2. Find it and use it to approximate π^2 with a number correct to 3 digits. How many terms were needed? (You will need a technology tool like ALAMA calculator for this problem.