

# JDEP 384H: Numerical Methods in Business

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110 Kaufmann Center

# Outline

- 1 Basic Financial Assets and Related Issues
- 2 A Peek at Optimization Theory: Linear Programming

## Example

A coupon bond has face value of \$100, rate of 6%, period of six months and maturity of three years. Price the bond at a yield (=discount rate) of 7% and find the change in price if yield is increased to 7.5%. Compare to the first and second-order (quadratic) approximation at  $\lambda = 0.07$ .

**Solution.** Let's do it with Matlab, using the functions `pvvar`, `cfdrm` and `cfconv`. Here's a start:

```
>cf = [0, 3, 3, 3, 3, 3, 103], cfp = cf(2:length(cf))
>lam = 0.07/2
>deltalam = 0.005/2
>P = pvvar(cf,lam)
>deltap = pvvar(cf,lam+deltalam)-P % exact change in P
>% linear and quadratic approximations to change in P
>deltap1 = -cfdrm(cfp,lam)*P*deltalam
>deltap2 = deltap1+P*cfconv(cfp,lam)*deltalam^2/2
>% now repeat with deltalamb = -0.005/2
```

# Postscript on Bond Pricing

One outcome of the previous discussion is that the duration and convexity of a “dirty” bond are the same as what one would get by using the clean price and the bond it represents to calculate duration and convexity.

Reason:

$$\frac{d}{d\lambda} P_{dirty} = \frac{d}{d\lambda} (P_{clean} + \text{prorate} * \text{Coupon}) = \frac{d}{d\lambda} P_{clean}.$$

# A Generic Problem

## Problem:

We have a stream of known liabilities in the future and want to create a portfolio of bonds (from a given set of bonds) such that the income stream of these bonds will help us meet future liabilities, and at a minimal cost.

- Want to avoid the difficulties of rate changes (thus, avoid reinvestment risk.)
- Outrageous but useful for calculation assumption: yield is the same across all bonds.
- Also assume: we only deal in risk-free bonds.

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# Immunitization

## Basic Idea:

Set up our bond portfolio in such a way that its sensitivity to yield fluctuations approximates that of the liability stream.

- How? Create our bond portfolio in such a way that (modified) duration (and possibly convexity) of portfolio of liabilities match corresponding quantities in the liability portfolio at a given  $\lambda$ .
- For a *single* bond, this is easy. For a portfolio of bonds take weighted averages to represent the duration (convexity) of the portfolio.
- For two or more bonds, take weighted averages to represent the duration (convexity) of the portfolio.
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## Example

You have two available instruments: a 5-year bond with coupon rate of 8% and face value \$100 and a 1-year coupon bond of the same face value and coupon rate 6%. Both give annual coupons. Call the bonds A and B, respectively. Prices are determined by the prevailing interest rate, which is currently 6%. Your objective: to make \$100,000 at the end of 3 years, at which point you will cash in your portfolio. You make an initial investment of \$83,962, which is the present value of your objective at the current prevailing rate

**Solution.** We'll use Matlab for these calculations. Examine the script `Immuredur.m` and run it.

## Further notes:

- Yields may be different, maturity dates may differ, etc., so the weighted durations and convexities are only approximations. However, if all bonds had the same yield, then these weighted averages would give the correct duration and convexity for the portfolio.
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## To fix ideas:

- We have a portfolio of three bonds whose duration/convexities we have calculated as  $D_i, C_i, i = 1, 2, 3$ , and we want to shape the portfolio so as to match duration  $D$  and convexity  $D$ . How to immunize?
- Say that  $w_1, w_2, w_3$  represent the weights of each of the three bonds in our portfolio (so  $w_1 + w_2 + w_3 = 1$ ).
- **First Pass:** Get a perfect match. This gives a linear system of three equations in three unknowns. Write them out at the board! But: solutions may have negative weights!!

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- **Second Pass:** Disallow short positions and add more bonds to the portfolio for flexibility. Now there are too many possibilities. So we make a choice among all possibilities by optimizing a cost or profit functional. For example, we could maximize the weighted yield.
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# Preliminary Notions

See pp. 366 of BT for further discussion. Also, our LinearAlgebraLecture notes has a concise summary of key definitions and properties.

## Example:

Sketch the solution set to the inequality sets given below, where in all cases it is understood that all variables  $x_i$  satisfy the inequality  $x_i \geq 0$ .

- $x_1 + x_2 \leq 3$ .
- Preceding plus  $2x_1 + x_2 \leq 4$ .
- Preceding two plus  $2x_1 + 3x_2 \geq 6$ .
- Change last inequality to  $2x_1 + 3x_2 \geq 12$ .

Examine LinearAlgebraLecture-384H.pdf after these concrete examples and calculations.

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