

ASSIGNMENT 2 FOR JDEP 384H

**Thomas Shores**  
**Department of Mathematics**  
**University of Nebraska**  
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Points: 45

Due: February 15

1. (8 pts) In this exercise you are given that three securities  $S_1, S_2, S_2$ , have expected rates of return  $\mu_1 = 0.1, \mu_2 = 0.15$  and  $\mu_3 = 0.2$ , respectively. The standard deviation of the securities are  $\sigma_1 = 0.28, \sigma_2 = 0.24$  and  $\sigma_3 = 0.25$ , respectively. The correlations between these rates of return are  $\rho_{12} = -0.1, \rho_{23} = 0.2$  and  $\rho_{31} = 0.25$ . Define the vector random variable  $\mathbf{S} = (S_1, S_2, S_3)$ . You may find it helpful to review the section in ProbStatLecture-384H on vector random variables.

(a) Find the mean vector and covariance matrix for  $\mathbf{S}$ .

(b) If one owns a portfolio which is 50% invested in  $S_1$  and 25% in each of the remaining two, what is the expected return and standard deviation of this portfolio?

(c) What is the weighting of the portfolio that yields the maximum expected return? What is the corresponding risk (as measured by the standard deviation of this portfolio)?

(d) Find a weighting with no short positions that is also an eigenvector of the covariance matrix of  $S$ . What are the risk (as measured by variance) and expected return of this portfolio?

**Solution.** Here is a diary file of our calculations:

```
octave:2> mystartup
octave:3> mu = [0.1; 0.15; 0.2] % (a) mean vector
mu =
0.10000
0.15000
0.20000
octave:4> D = diag([0.28,0.24,0.25]) % standard deviation diagonal
D =
0.28000 0.00000 0.00000
0.00000 0.24000 0.00000
0.00000 0.00000 0.25000
octave:5> R = [1 -0.1 0.25; -0.1 1 0.2; 0.25 0.2 1] % correlation matrix
R =
1.00000 -0.10000 0.25000
-0.10000 1.00000 0.20000
0.25000 0.20000 1.00000
octave:6> S = D*R*D % (a) covariance matrix
S =
0.0784000 -0.0067200 0.0175000
-0.0067200 0.0576000 0.0120000
0.0175000 0.0120000 0.0625000
octave:7> wts = [0.5; 0.25; 0.25] % (b)
wts =
0.50000
```

```

0.25000
0.25000
octave:8> wts'*mu % expected return
ans = 0.13750
octave:9> sqrt(wts'*S*wts) % standard deviation
ans = 0.17692
octave:10> (c) requires no calculation: put all eggs in highest return
octave:10> wts = [0 0 1]'
wts =
0
0
1
octave:11> % (c) risk is easy, too
octave:11> risk = D(3,3)
risk = 0.25000
octave:12> [evecs,evals] = eig(S) % (d) find evecs and evals
evecs =
0.414856 0.363246 -0.834234
0.622233 -0.782211 -0.031164
-0.663868 -0.506159 -0.550529
evals =
0.040317 0.000000 0.000000
0.000000 0.068486 0.000000
0.000000 0.000000 0.089698
octave:13> wts = evecs(:,3)/sum(evecs(:,3) ) % (d) the good eigenvector
wts =
0.589179
0.022010
0.388812
octave:14> risk = sqrt(wts'*S*wts) % risk of this portfolio
risk = 0.21152
octave:15> returns = wts'*mu % return of this portfolio
returns = 0.13998

```

2. (10 pts) The data in the text file `TreasuryRates2003` represents the weekly values of the 3-Month, 6-Month, 1-Year, 2-Year and 3-Year TCM (Treasury Constant Maturity) notes, one row per week. You can obtain this variable by the Matlab command “load `TreasuryRates2003`”, assuming this data is in your current working directory.

(a) View each row as a sample from the 5-vector random variable  $\mathbf{X}$  and use Matlab to compute the mean and covariance matrix of  $\mathbf{X}$ . (Use commands `cov` and `mean`.)

(b) Make a plot of the means of the rates of the notes against maturity (in years). What does this term structure suggest to you about the future market?

(c) Find the correlation matrix for  $\mathbf{X}$  and find the notes that have the strongest correlation and weakest correlations.

(d) Treat the mean and variance for  $X_5$  (3-Year notes) over the year’s data as the true mean and variance. Now compute confidence intervals for the true mean using the the first 26 weeks of data, assuming (i) known variance and (ii) unknown variance, and also find a confidence interval for the true variance. You may assume the samples are i.i.d. random normal variables. Do your intervals contain the “true” mean and variance?

**Solution.** Here is a diary file of our calculations with comments, followed by the graph generated:

```

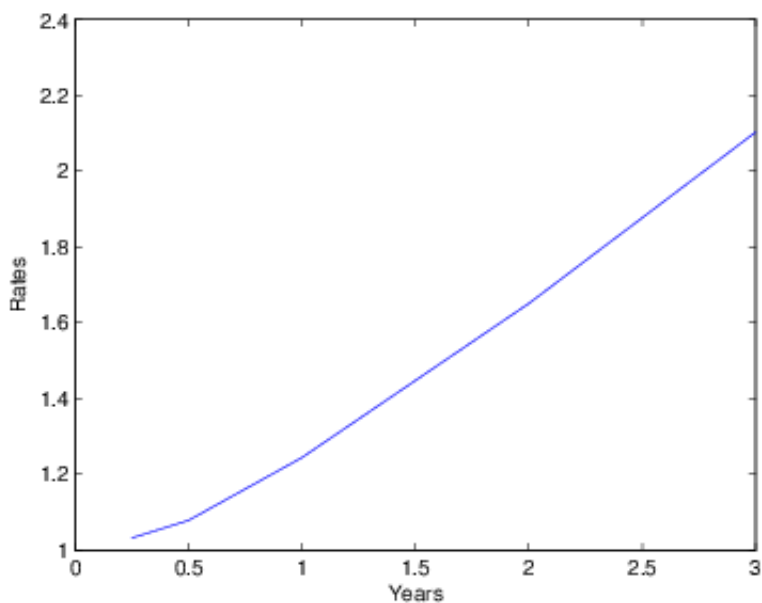
mystartup
load TreasuryRates2003
% (a) covariance matrix
S = cov(TreasuryRates2003)
S =
0.0131 0.0103 0.0048 -0.0016 -0.0050
0.0103 0.0088 0.0063 0.0039 0.0031
0.0048 0.0063 0.0114 0.0204 0.0264
-0.0016 0.0039 0.0204 0.0475 0.0645
-0.0050 0.0031 0.0264 0.0645 0.0896
% (a) find mean
mu = mean(TreasuryRates2003)'
mu =
1.0313
1.0779
1.2437
1.6508
2.1035
% (b) year axis
years = [0.25; 0.5; 1; 2; 3]
years =
0.2500
0.5000
1.0000
2.0000
3.0000
plot(years,mu)
xlabel('Years')
ylabel('Rates')
% (c) find std. dev. diagonal
D = diag(sqrt(diag(S)))
D =
0.1145 0 0 0 0
0 0.0938 0 0 0
0 0 0.1070 0 0
0 0 0 0.2179 0
0 0 0 0 0.2994
% (c) correlation matrix
R = inv(D)*S*inv(D)
R =
1.0000 0.9561 0.3930 -0.0641 -0.1453
0.9561 1.0000 0.6240 0.1908 0.1113
0.3930 0.6240 1.0000 0.8766 0.8247
-0.0641 0.1908 0.8766 1.0000 0.9888
-0.1453 0.1113 0.8247 0.9888 1.0000
% (c) The strongest correlation is between 2- and
% 3- year bonds (X_4 and X_5). The weakest correlation
% is between 3-month and 2-year bonds (X_1 and X_4)

```

```

(d) find "true" variance of X_5
variance5 = S(5,5)
variance5 =
0.0896
% (d) find "true" mean of X_5
mean5 = mu(5)
mean5 =
2.1035
% sample mean of 1st 26 weeks
samplemean5 = mean(TreasuryRates2003(1:26,5))
samplemean5 =
1.9254
% sample covariance of 1st 26 weeks
samplevar5 = var(TreasuryRates2003(1:26,5))
samplevar5 =
0.0594
z95 = norm_inv(0.05/2,0,1) % let's go for 95% confidence interval
z95 =
-1.9600
% compute confidence interval with sample mean and "known" variance
confidenceinterval = [samplemean5 + z95*sqrt(variance5/26),samplemean5 - z95*sqrt(variance5/26)
confidenceinterval =
1.8103 2.0405
% (d) so the "true mean" mean5 = 2.1035 falls outside this interval
% next, try it with unknown variance and t-statistics at 95%
t95 = tdis_inv(0.05/2,25)
t95 =
-2.0595
confidenceinterval = [samplemean5 + t95*sqrt(samplevar5/26),samplemean5 - t95*sqrt(samplevar5/26)
confidenceinterval =
1.8269 2.0238
% (d) again, the "true mean" mean5 = 2.1035 falls outside this interval
% finally, get a confidence interval for the variance
confidenceintervalvar = [25*samplevar5/chis_inv(1-0.05/2,25), 25*samplevar5/chis_inv(0.05/2,25)
confidenceintervalvar =
0.0365 0.1132
% (d) this time "true variance", variance5 = 0.0896, falls in this confidence interval

```



This plot of an increasing term rate structure suggests that higher inflation is anticipated, as happens during a growth period.

**3.** (10 pts) Refer to Lecture 9 notes for the Wyndor Glass Company problem.

(a) Solve this problem graphically.

(b) Solve it using the Matlab programs `linprog` and `lp`.

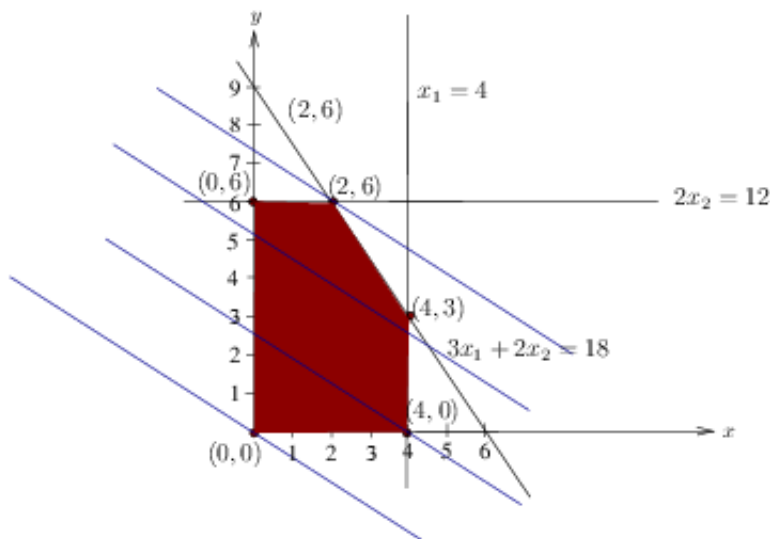
(c) Suppose the profit per batch is changed from \$3K, \$5K to \$3K, \$2K. Solve this problem graphically. How does it differ from the solution in part (a)?

(d) Suppose management insists that Plant 3 must run at full capacity (no slack time). Solve this modified problem graphically.

**Solution.** (a) The problem can be expressed as

$$\begin{aligned}
 \text{Max } Z &= 3x_1 + 5x_2 \\
 \text{Sbj} \\
 x_1 &\leq 4 \\
 2x_2 &\leq 12 \\
 3x_1 + 2x_2 &\leq 18 \\
 x_j &\geq 0, \quad j = 1, 2, 3.
 \end{aligned}$$

Here each  $\leq$  constraint represents utilization of a plant's capacity and  $Z$  is the total profit. The graph:



Here the feasible set is colored. The level curves  $Z = 3x_1 + 5x_2$ ,  $Z$  constant, move to the right as  $Z$  increases, so the maximum occurs at the corner  $(2,6)$ , where the value of  $Z$  is  $Z = 3 \cdot 2 + 5 \cdot 6 = 36$ .

(b) Here is a transcript of the solution by lp and linprog. The latter requires equality.

```

octave:2> mystartup
octave:3> % set up for using linprog
octave:3> c = [3;5]
c =
 3
 5
octave:4> A = [1 0;0 2;3 2]
A =
 1 0
 0 2
 3 2
octave:5> b = [4 12 18]
b =
 4
 12
 18
octave:6> [xopt,zmin] = linprog(-c,A,b) % we must turn max into min
xopt =
 2
 6
zmin = -36
octave:7> % for lp we need equalities, so add slack variables to c and A
octave:7> [xopt,zmin] = lp (-[c;zeros(3,1)], [A,eye(3)],b)
xopt =
 2
 6
 2
 0

```

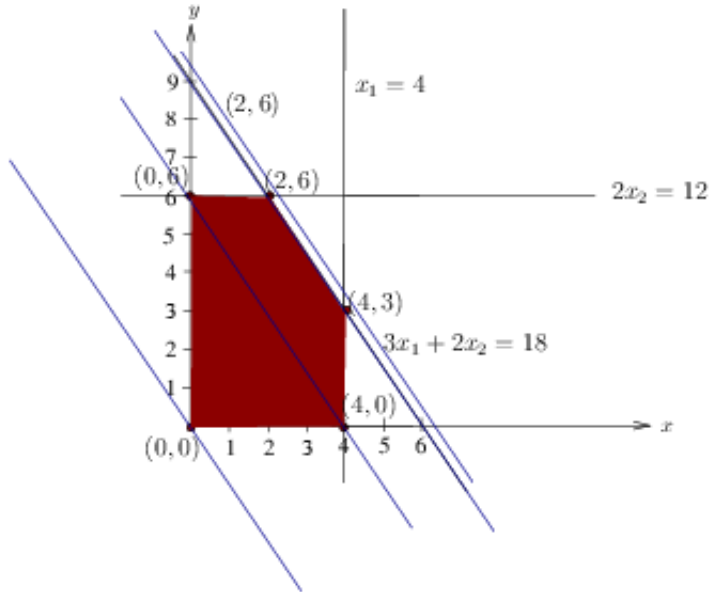
0

zmin = -36

octave:8&gt; quit

From this we see that the actual optimum profit is  $Z = 36$  and it occurs at the point  $x_1 = 2, x_2 = 6$ .

(c) Here is the graphical solution:

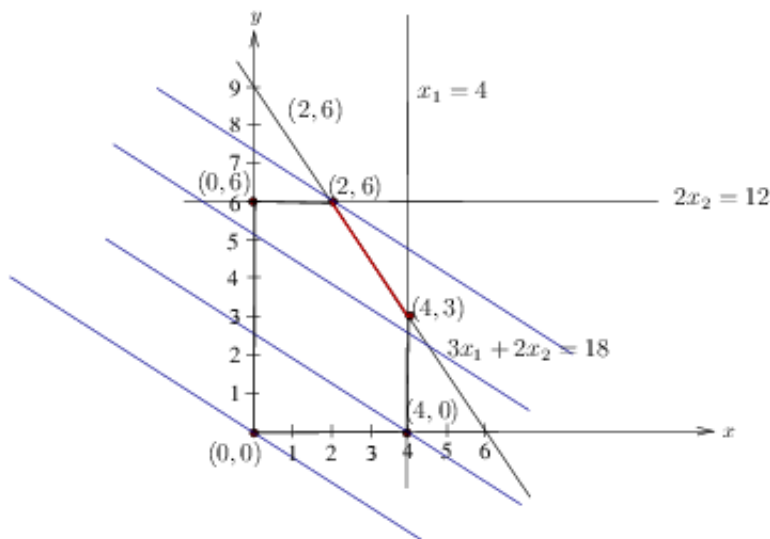


Here the level curves  $Z = 3x_1 + 2x_2$ ,  $Z$  constant, move to the right as  $Z$  increases, so the optimum occurs along the entire segment from the point  $(4, 3)$  to  $(2, 6)$  on the constraint line  $3x_1 + 2x_2 = 18$ . The value of the maximum is  $Z = 18$ .

(d) The problem can be expressed as

$$\begin{aligned} \text{Max } Z &= 3x_1 + 5x_2 \\ \text{Sbj} \\ x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &= 18 \\ x_j &\geq 0, \quad j = 1, 2, 3. \end{aligned}$$

The graph:



We see from this that the feasible set in this case is the line segment from the point  $(2, 6)$  to  $(4, 3)$ . However, the objective function is not changed from (a), so the level curves for  $Z$  are the same as in (a) and the optimum profit occurs at corner  $(2, 6)$ , where the value of  $Z$  is  $Z = 3 \cdot 2 + 5 \cdot 6 = 36$ .

4. (9 pts) The future value  $S_1$  of a stock in one time period with present value  $S_0 = 100$  is modeled by a binomial lattice with multiplicative shocks of  $u = 1.2$  and  $d = 0.8$ . Probability of up is  $p$  and down is  $1 - p$ , but  $p$  is not known. A zero-coupon bond with face value  $B_0 = 100$  earns risk-free interest 10% in the same time period. You construct a weighted portfolio of  $x$  stocks and  $y$  bonds (short positions in either asset are allowed.)

(a) What is the value of your portfolio in terms of  $x$  and  $y$  in one time period in the cases that the stock goes up or down?

(b) What is the value of a call option (at strike price  $S_0$ ) on the stock in one time period in the cases that the stock goes up or down?

(c) Match the values of (a) and (b) in the cases that the stock goes up or down, and solve the resulting system of equations for  $x$  and  $y$ . What is the present price of this portfolio?

(d) With the standard market assumptions, the present price of this portfolio is exactly the present market price of a call option on the stock. Explain why.

**Solution.** (a) The value of the portfolio in one time step will be

$$V = \begin{cases} 120x + 110y & \text{if stock goes up} \\ 80x + 110y & \text{if stock goes down} \end{cases}$$

(b) If the price of the stock goes up, the value of the call is \$20, since we could borrow \$100, buy the stock at that strike price, then immediately sell it for \$120, repay the loan and keep \$20. Of course, we are ignoring transaction costs.

If the price of the stock goes down, the value of the call option is zero, since exercising it would require one to buy for \$100 a stock that is worth \$80.

(c) Matching the up and down prices of option and portfolio gives the two equations

$$120x + 110y = 20$$

$$80x + 110y = 0.$$

These are easily solved to obtain that

$$x = \frac{1}{2}$$

$$y = -\frac{4}{11}.$$

The present price of this portfolio is

$$P = \frac{1}{2}100 - \frac{4}{11}100 = \frac{1}{2} - \frac{4}{11} \approx 13.636.$$

(d) The reason that the present price  $P$  of the portfolio must be the price  $C$  of the option is as follows: Since the market is liquid, we can build a portfolio as described in (c).

- If  $C > P$ , then we should sell one call at price  $C$  and use the proceeds to buy one share of the portfolio described above (this is possible since we assume the market is liquid, so assets may be bought and sold in any amount), and pocket  $C - P$ . At expiry the price of our portfolio matches the price of the call in both cases, so we sell our portfolio and use the proceeds to pay the caller. Thus we come out with  $C - P$ , which is arbitrage and not permissible.
- If  $C < P$ , then we should sell one unit of the portfolio at price  $P$  and use the proceeds to buy one call and pocket  $P - C$ . At expiry, again the price of the portfolio matches the price of the call, so we should sell (i.e., exercise) the call and use the proceeds to repay the portfolio. This time we get to pocket  $P - C$ , which is arbitrage and not permissible.

We conclude that we must have  $P = C$ .

5. (10 pts) Suppose that you have the same objective and tools as in the example on page 14 of Lecture 8, except that you have additionally a three year coupon bond C with annual coupons of 7.5% and face value of \$100. Also, you disallow short positions in any bond. For your convenience, a modification of `ImmuneDur.m` with name `Exercise2_5.m` is to be found in our home directory.

(a) Is there a weighting that matches weighted durations and convexities to the duration and convexity of the target cash flow?

(b) Find the weighted combination of the three bonds that matches duration and maximizes the convexity of the portfolio.

(c) Find the weighted combination of the three bonds that matches duration and maximizes the weighted coupon rate of the portfolio.

(d) Calculate the displacement from your objective if the prevailing interest rate rises or falls 3% after one year for each viable strategy above. Which is superior in this case?

**Solution.** First, we list the script that did the calculations and its output when run. Comments follow for each problem part.

```
% Script: Exercise2_5Key.m
% description: this file calculates some of the
% the numbers required by Exercise 2.5.
% We have simply modified the file Exercise2_5.m.
% Before we do the parts, we need a few numbers.
% We eliminate some of the verbiage of ImmuneDur.m
disp('Calculate the numbers for each bond separately under the rate change scenarios.')
```

```

r = 0.06;
goal = 100000;
invest = goal/(1+r)^3;
bondA = [0 8 8 8 8 108];
priceA = pvvar(bondA,r);
couponA = bondA(2);
bondB = [0 106];
priceB = pvvar(bondB,r);
couponB = 6;
bondC = [0 7.5 7.5 107.5];
priceC = pvvar(bondC,r);
couponC = bondC(2);
deltar = 0.03;
finalcashA = zeros(2,1);
numbondA = invest/pvvar(bondA,r);
newr = r + deltar;
moneysofar = numbondA*couponA*((1+newr)*(1+newr) + (1+newr) + 1);
bondsaleprice = pvvar([0,bondA(5:6)],newr);
finalcashA(1) = moneysofar + numbondA*bondsaleprice;
newr = r - deltar;
moneysofar = numbondA*couponA*((1+newr)*(1+newr) + (1+newr) + 1);
bondsaleprice = pvvar([0,bondA(5:6)],newr);
finalcashA(2) = moneysofar + numbondA*bondsaleprice;
finalcashB = zeros(2,1);
newr = r + deltar;
moneysofar=invest*(1+r)*(1+newr)^2;
finalcashB(1)=moneysofar;
newr = r - deltar;
moneysofar=invest*(1+r)*(1+newr)^2;
finalcashB(2)=moneysofar;
finalcashC = zeros(2,1);
numbondC = invest/pvvar(bondC,r);
newr = r + deltar;
moneysofar = numbondC*couponC*((1+newr)*(1+newr) + (1+newr) + 1);
finalcashC(1) = moneysofar + numbondC*100;
newr = r - deltar;
moneysofar = numbondC*couponC*((1+newr)*(1+newr) + (1+newr) + 1);
finalcashC(2) = moneysofar + numbondC*100;
disp('(a) Immunize the portfolio by matching durations and convexity.')
```

disp('Target duration is like a zero coupon bond with face value \$100,000:')

```

targetdur = cfdur([0 0 100000],0.06);
targetconv = cfconv([0 0 100000],0.06);
durA = cfdur(bondA(2:length(bondA)),0.06);
durB = cfdur(bondB(2:length(bondB)),0.06);
durC = cfdur(bondC(2:length(bondC)),0.06);
convA = cfconv(bondA(2:length(bondA)),0.06);
convB = cfconv(bondB(2:length(bondB)),0.06);
convC = cfconv(bondC(2:length(bondC)),0.06);
coef = [durA durB durC; convA convB convC; 1 1 1];
rhs = [targetdur; targetconv; 1];
```

```

wtsa = coef\rhs
disp('(b) Use linprog to match duration and maximize convexity.')
wtsb = linprog(-[convA; convB; convC], [], [], [], [], [1 1 1; durA, durB, durC], [1; targetdur])
disp('(c) Use linprog to match duration and maximize weighted coupon rate.')
wtsc = linprog(-[0.08; 0.06; 0.075], [], [], [], [], [1 1 1; durA, durB, durC], [1; targetdur])
finalcash = zeros(2,1);
finalcash = [finalcashA, finalcashB, finalcashC]*wtsb;
disp(' Displacement from our goal of $100,000 with weighting wtsb:')
deltafinalcash = finalcash - [100000; 100000]
finalcash = [finalcashA, finalcashB, finalcashC]*wtsc;
disp(' Displacement from our goal of $100,000 with weighting wtsc:')
deltafinalcash = finalcash - [100000; 100000]

```

Output:

```

octave:17> Exercise2_5Key
Calculate the numbers for each bond separately under the rate change scenarios.
(a) Immunize the portfolio by matching durations and convexity.
Target duration is like a zero coupon bond with face value $100,000:
wtsa =
0.011533
-0.101369
1.089836
(b) Use linprog to match duration and maximize convexity.
wtsb =
0.59840
0.40160
0.00000
(c) Use linprog to match duration and maximize weighted coupon rate.
wtsc =
0.12981
0.00000
0.87019
Displacement from our goal of $100,000 with weighting wtsb:
deltafinalcash =
141.36
149.61
Displacement from our goal of $100,000 with weighting wtsc:
deltafinalcash =
28.429
30.218
octave:18> quit

```

(a) From the three equations we see that the weighting is uniquely determined and calculation above shows that  $w_2$  must be negative. Since we disallow short positions, there is no admissible weighting that matches convexity and duration.

(b) We use `linprog.m` to find `wtsb` for this part. The problem is to minimize  $-0.08w_1 - 0.06w_2 - 0.075w_3$  subject to the constraints  $w_1 + w_2 + w_3 = 1$  and  $D_A w_1 + D_B w_2 + D_C w_3 = 3$ , where the  $D$ 's are durations.

(c) We use `linprog.m` to find `wtsb` for this part. The problem is to minimize  $-C_A w_1 - C_B w_2 - C_C w_3$  subject to the constraints  $w_1 + w_2 + w_3 = 1$  and  $D_A w_1 + D_B w_2 + D_C w_3 = 3$ , where the  $C$ 's are convexities.

(d) From the output of the script, we see that the smaller, hence superior, displacements follow from using `wtsc`. Hence maximizing weighted coupon rate does best in this case.