ASSIGNMENT 1 FOR JDEP 384H

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Points: 45
Due: January 25

**1.** (5 pts) Let

$$A = \begin{bmatrix} 3 & -1 & 0 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix} \text{ and } x = [1, -2, -1, 3].$$

Perform the following calculations by hand (use Matlab to check your work, if you're not sure of the answers) or explain why they are impossible:
(a) $A\mathbf{x}^T$ (b) $AA^T$ (c) $AA^T\mathbf{x}^TA$
(d) $A + I_2\mathbf{x}$ (e) $I_2A$ (f) $\|\mathbf{x}\|_2$
(g) $\|\mathbf{x}\|_1$ (h) $\|\mathbf{x}\|_\infty$

**Solution.** (a) $A\mathbf{x}^T = \begin{bmatrix} 3 & -1 & 0 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix}$

(b) $AA^T = \begin{bmatrix} 3 & -1 & 0 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 11 \\ 11 & 21 \end{bmatrix}$

(c) Not possible since $\mathbf{x}^T$ is $1 \times 4$ and $A$ is $2 \times 4$.
(d) Not possible since $I_2\mathbf{x}$ is undefined ($\mathbf{x}^T$ is $1 \times 4$ and $I_2$ is $2 \times 2$).
(e) $I_2A = A$
(f) $\|\mathbf{x}\|_2 = \sqrt{1^2 (-2)^2 + (-1)^2 + |3|^2} = \sqrt{15}$\pi/6
(g) $\|\mathbf{x}\|_1 = |1| + |-2| + |-1|, + |3| = 7$
(h) $\|\mathbf{x}\|_\infty = \max\{|1|, |-2|, |-1|, |3|\} = 3$

**2.** (8 pts) Consider the following linear system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 &= 2 \\ 2x_1 - x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

(a) Exhibit the coefficient matrix $A$ and right hand side vector $\mathbf{b}$ for this system, so that if $\mathbf{x} = (x_1, x_2, x_3, x_4)$, then the system is given by $A\mathbf{x} = \mathbf{b}$. **Note:** I use the convention that a vector given by soft parentheses represents a column vector. Therefore

$$(x_1, x_2, x_3, x_4) = [x_1, x_2, x_3, x_4]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

(b) Get help on the Matlab backslash command \ and use it to solve $A\mathbf{x} = \mathbf{b}$.
(c) Compute the residual vector $\mathbf{b} - A\mathbf{x}$. Is it the zero vector?
(d) Write out explicitly the linear system defined by the matrix equation

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

(e) Calculate the norm of the residual

$$\mathbf{r} = A^T\mathbf{b} - A^T A\mathbf{x}$$

where $\mathbf{x}$ is the vector obtained in (b). Conclusions?
**Solution.** (a) The coefficient matrix of this system is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

and right-hand side is $\mathbf{b} = (3, 2, 1, 5)$.

(b), (c) Here is the residual calculation in Matlab/Octave:

```
octave:2> A = [1 1 1;1 1 0; 2 -1 0;1 2 1]
A =
1 1 1
1 1 0
2 -1 0
1 2 1
octave:3> b = [3,2,1,5]'
b =
3
2
1
5
octave:4> x = A\b
x =
1.0435
1.2174
1.1304
octave:5> residual = b - A*x
residual =
-0.39130
-0.26087
0.13043
0.39130
```

Clearly, the residual is *not* zero.

(d) The linear system $A^T A\mathbf{x} = A^T\mathbf{b}$ question is

$$\begin{aligned} 7x_1 + 2x_2 + 2x_3 &= 12 \\ 2x_1 + 7x_2 + 3x_3 &= 14 \\ 2x_1 + 3x_2 + 2x_3 &= 8 \end{aligned}$$

(e) The residual is given by

```
octave:7> A'*b-A'*A*x
ans =
0.0000e+00
-1.7764e-15
-1.7764e-15
```

Evidently, the residual is numerically zero, which means that the solution to the problem found by Matlab or Octave is actually the solution the the normal equations, which is therefore the least squares solution to the problem.

**3.** (8 pts) Recall (or accept) that a plane rotation matrix is defined by the formula

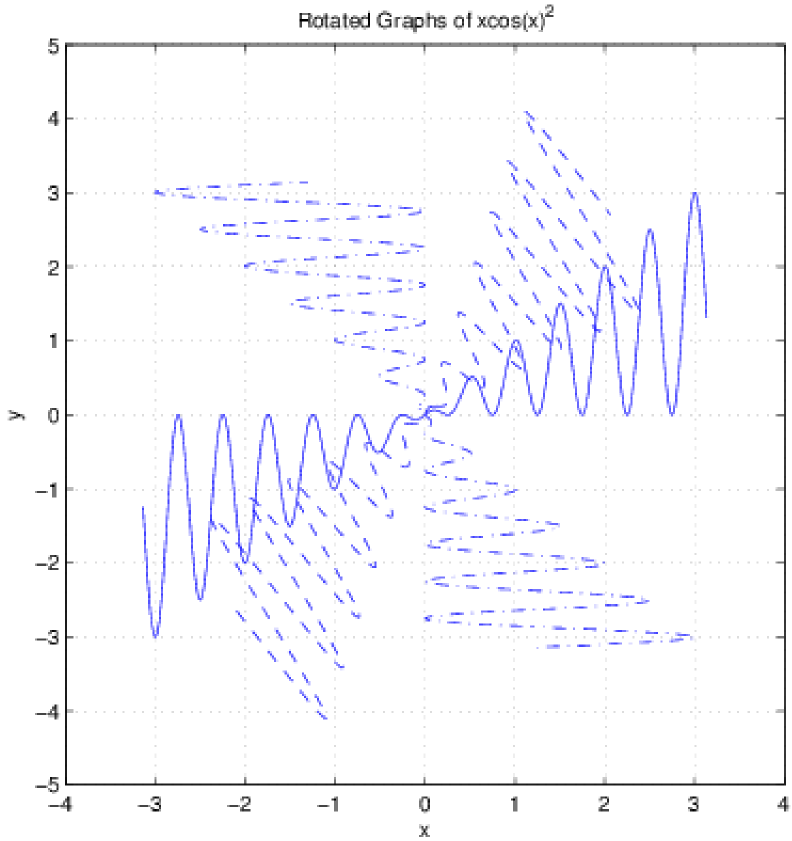
$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

If $\mathbf{x} = (x, y)$ represents a point in the plane, then $R(\theta)\,\mathbf{x}$ represents the point obtained by rotating $\mathbf{x}$ counterclockwise about the origin by an angle of $\theta$ radians.
(a) Use Matlab to plot the function $y = x\cos^2(2\pi x)$ on the interval $[-\pi, \pi]$.
(b) Set hold on and superimpose plots of this curve rotated by both $\pi/6$ and $\pi/2$ radians. (Hint: it helps to form the matrix $[x; y]$ whose columns are points of the plot. Then multiply by the appropriate rotation matrix. The new rows are what you should plot.)
**Solution.** (a) and (b) Here is a transcript of a Matlab session that did it:

```
mystartup
  x = -pi:.01:pi;
  y = x.*cos(2*pi*x).^2;
  theta = pi/6
  theta =
  0.5236
  R = [cos(theta),-sin(theta); sin(theta),cos(theta)]
  R =
  0.8660 -0.5000
  0.5000 0.8660
  plot(x,y)
  grid,hold on
  X = [x;y]; % matrix of coordinates
  X1 = R*X; % rotated coordinates
  plot(X1(1,:),X1(2,:))
  theta = pi/2
  theta =
  1.5708
  R = [cos(theta),-sin(theta); sin(theta),cos(theta)]
  R =
  0.0000 -1.0000
  1.0000 0.0000
  X2 = R*X; % rotated coordinates
  plot(X2(1,:),X2(2,:))  % pretty up graphic with gui and save it as eps
```

```
quit
```

The resulting graph follows.




**4.** (8 pts) Let

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 4 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

(a) Think of $A$ as a transition matrix for a discrete dynamical system and find the state vector $\mathbf{x}^{(20)}$, starting with a random initial vector (use Matlab's rand command). Then calculate $\mathbf{x}^{(20)} / \left\| \mathbf{x}^{(20)} \right\|_2$.
(b) Next use Matlab's eig command to compute an eigensystem for $A$ (eigenvectors plus eigenvalues).
(c) Comment on any connections you see between parts (a) and (b).
**Solution.** (a) and (b). An Octave script that does it

```
octave:2> A = [1 4 -1; 4 3 0; -1 0 2]
  A =
  1 4 -1
  4 3 0
  -1 0 2
  octave:3> x = randn(3,1)
  x =
```

```
-0.62312
-1.82095
-0.95608
octave:4> x20 = A^20*x
x20 =
-7.5461e+15
-9.3898e+15
1.7905e+15
octave:5> x20 = x20/norm(x20)
x20 =
-0.61962
-0.77101
0.14702
octave:6> [P,D] = eig(A)
P =
0.783042 -0.054009 0.619620
-0.594322 0.228744 0.771010
0.183376 0.971987 -0.147018
D =
-2.27015 0.00000 0.00000
0.00000 2.05557 0.00000
0.00000 0.00000 6.21459
```

(c) It is clear from inspection that $\mathbf{x}^{(20)}$ is equal, to the displayed digits, to the third eigenvector calculated by Matlab, which corresponds to the third and largest eigenvalue, $\lambda = 6.21459$. We calculate that

```
octave:7> x20 + P(:,3)
ans =
-2.1147e-09
1.5785e-09
-6.3424e-10
```

This confirms our suspicion.

**5.** (8 pts) Let $f(x) = \frac{1}{1+x^2} + \cos(2x)$ be defined on the interval $[0, 6]$. In this exercise you will be asked to create multiple plots together in a single figure.

(a) Compute the function $f'(x)$.

(b) Create a figure with four subplots. To find out how to do so, read the last page of the file MatlabLecture-384H.pdf.

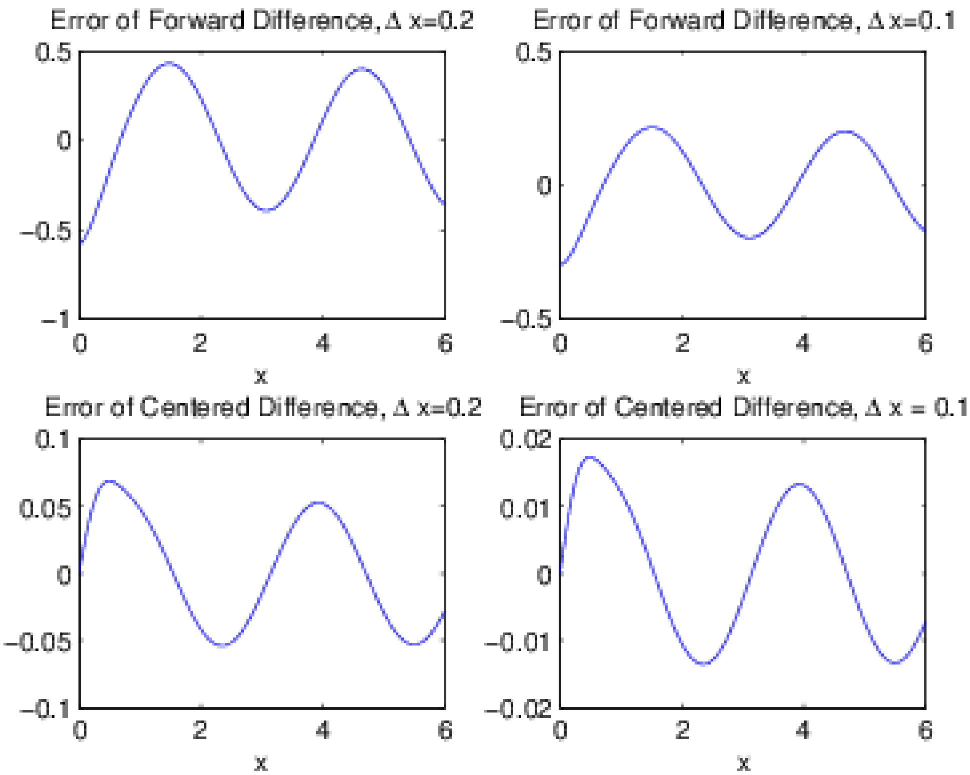
(c) Use the forward difference $(f(x + \Delta x) - f(x)) / \Delta x$ as an approximation to $f'(x)$ and insert into the figure plots of the difference $(f(x + \Delta x) - f(x)) / \Delta x - f'(x)$ on the interval $[0, 6]$ using $\Delta x = 0.2$ and $\Delta x = 0.1$.

(d) The centered difference $(f(x + \Delta x) - f(x - \Delta x)) / (2\Delta x)$ is supposed to be an improved approximation to the derivative. Repeat (c) with entered difference in place of forward difference and comment on the differences in the graphs that you see.

**Solution.** (a) By calculus we obtain that

$$
\begin{aligned}
f'(x) &= \left(\frac{1}{1+x^2}\right)' + (\cos(2x))' \\
&= -1\left(1+x^2\right)^{-2} 2x + - \sin(2x)\, 2 \\
&= \frac{-2x}{(1+x^2)^2} - 2\sin(2x)\,.
\end{aligned}
$$

(b), (c) and (d) Here is the picture we obtained. The picture generated by these Matlab commands was enhanced with some gui fiddling, resulting in




Here is the script that we used:

```
f = @(x) 1./(1+x.^2) + cos(2*x);
fp = @(x) -2*x./(1+x.^2).^2 -2*sin(2*x);
deltax = 0.2;
subplot(2,2,1)
x = 0:.01:6;
plot(x,(f(x+deltax)-f(x))/deltax - fp(x))
deltax = 0.1;
subplot(2,2,2)
plot(x,(f(x+deltax)-f(x))/deltax - fp(x))
deltax = 0.2;
subplot(2,2,3)
plot(x,(f(x+deltax)-f(x-deltax))/(2*deltax) - fp(x))
deltax = 0.1;
```

```
subplot(2,2,4)
plot(x,(f(x+deltax)-f(x-deltax))/(2*deltax) - fp(x))
% now pretty up the file with gui and save it as eps file
builtin('exit')
```

**Comment:** It is clear from the graphs that centered differences give better results. Furthermore, when we halve the spacing $\Delta x$ from 0.1 to 0.2 the error of forward differences retains the same general shape but halves in size, while when we use centered differences, the error reduces by almost a factor of 4.

**6.** (8 pts) Consult the common distributions subsection of the file ProbStatLecture-384H.pdf.
(a) Confirm the approximation assertion about Poisson vs binomial by calculating certain values, say with $\mu = 3, 4, 5$.
(b) Confirm the limiting assertion about the Student's t distribution by creating plots (in a single figure) of various t distributions and a standard normal distribution.
**Solution.** (a) The statement is that an application of the Poisson distribution is a "... limiting case of binomial distribution. Used, e.g., to approximate binomial distributions with large $n$ and constant $\mu = np$ of moderate size (typically $< 5$.)" To verify this we will take $\mu = 3, 4, 5$ and, for each choice of $\mu$, we take $n = 100$ and calculate the binomial distribution at equally space points in steps of 1. This distribution dies rapidly, so we only go to 10. Here is the script that generated the results and the resulting tables:

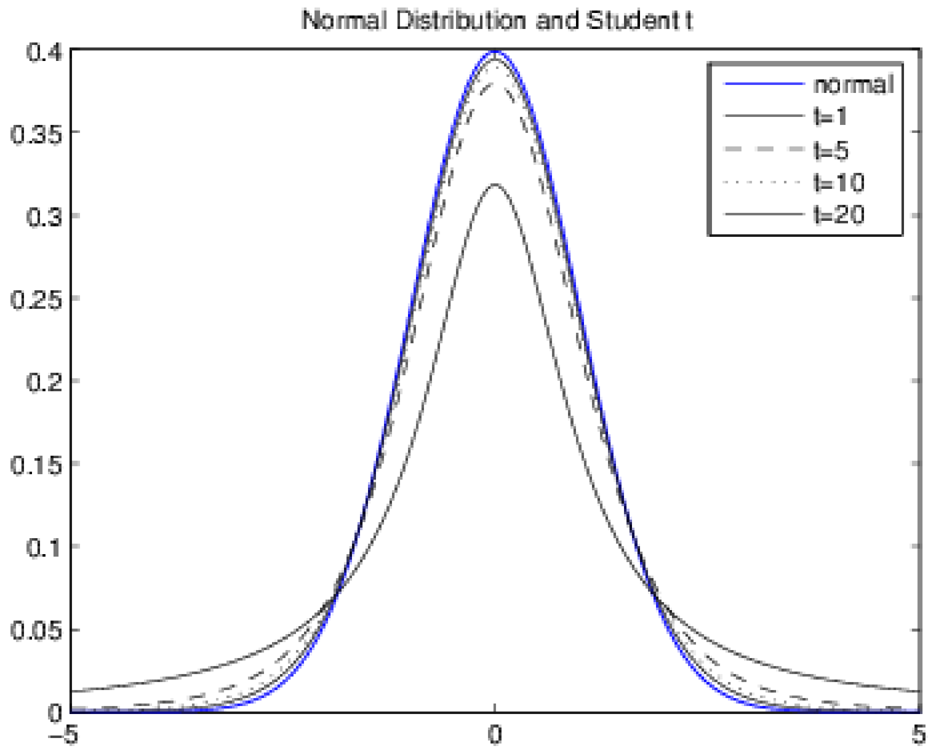
```
octave:2> mystartup
octave:3> x = (1:10)';
octave:4> n = 100;
octave:5> mu = 3
mu = 3
octave:6> table = zeros(10,3);
octave:7> table(:,1) = x;
octave:8> table(:,2) = bino_pdf(x,n,mu/n);
octave:9> table(:,3) = poisson_pdf(x,mu);
octave:10> disp(' x Binomial p.d.f.  Poison p.d.f.'),disp(table)
x Binomial p.d.f.  Poison p.d.f.
1.0000e+00 1.4707e-01 1.4936e-01
2.0000e+00 2.2515e-01 2.2404e-01
3.0000e+00 2.2747e-01 2.2404e-01
4.0000e+00 1.7061e-01 1.6803e-01
5.0000e+00 1.0131e-01 1.0082e-01
6.0000e+00 4.9610e-02 5.0409e-02
7.0000e+00 2.0604e-02 2.1604e-02
8.0000e+00 7.4078e-03 8.1015e-03
9.0000e+00 2.3420e-03 2.7005e-03
1.0000e+01 6.5913e-04 8.1015e-04
octave:11> mu = 4
mu = 4
octave:12> table(:,2) = bino_pdf(x,n,mu/n);
octave:13> table(:,3) = poisson_pdf(x,mu);
```

```
octave:14> disp(' x Binomial p.d.f.  Poison p.d.f.'),disp(table)
x Binomial p.d.f.  Poison p.d.f.
1.0000e+00 7.0293e-02 7.3263e-02
2.0000e+00 1.4498e-01 1.4653e-01
3.0000e+00 1.9733e-01 1.9537e-01
4.0000e+00 1.9939e-01 1.9537e-01
5.0000e+00 1.5951e-01 1.5629e-01
6.0000e+00 1.0523e-01 1.0420e-01
7.0000e+00 5.8880e-02 5.9540e-02
8.0000e+00 2.8520e-02 2.9770e-02
9.0000e+00 1.2147e-02 1.3231e-02
1.0000e+01 4.6059e-03 5.2925e-03
octave:15> mu = 5
mu = 5
octave:16> table(:,2) = bino_pdf(x,n,mu/n);
octave:17> table(:,3) = poisson_pdf(x,mu);
octave:18> disp(' x Binomial p.d.f.  Poison p.d.f.'),disp(table)
x Binomial p.d.f.  Poison p.d.f.
1.000000 0.031161 0.033690
2.000000 0.081182 0.084224
3.000000 0.139576 0.140374
4.000000 0.178143 0.175467
5.000000 0.180018 0.175467
6.000000 0.150015 0.146223
7.000000 0.106026 0.104445
8.000000 0.064871 0.065278
9.000000 0.034901 0.036266
10.000000 0.016716 0.018133
```

We can see that for $\mu = 3$, agreement is good to about $x = 5$, for $\mu = 4$ to about $x = 6$, and for $\mu = 5$ to about $x = 8$.

(b) The statement is that the Student's t-distrubition "approaches a standard normal distribution as $\nu \to \infty$. " To confirm this, we create plots of the standard normal distribution and the $t$-distribution for $t = 1, 5, 10, 15, 20$. This is done easily in Matlab or Octave, and here is the diary and resulting plot:

```
mystartup
x = (-5:0.01:5);
plot(x,norm_pdf(x,0,1))
hold on, grid
grid off
plot(x,tdis_pdf(x,1))
plot(x,tdis_pdf(x,5))
plot(x,tdis_pdf(x,10))
plot(x,tdis_pdf(x,20)) % we used gui to pretty up picture
```




Clearly, the $t$-distributions are approaching the standard normal distribution as $n$ gets large.