Solutions

Name:

Math 221 Section 5

Final Exam

Exams provide you, the student, with an opportunity to demonstrate your understanding of the techniques presented in the course. So:

Show all work. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) Find the general solution to the first order equation

$$x' = \frac{1}{t+1}x + \frac{2}{t+2}$$

$$x' - \frac{1}{t+1}x = \frac{2}{t+2} \qquad p(t) = \frac{-1}{t+1} \qquad g(t) = \frac{2}{t+2}$$

$$\int p(t)\,dt = \int \frac{-1}{t+1}\,dt = -\ln(t+1)$$

$$e^{\int p(t)dt} = e^{-\ln(t+1)} = e^{\ln\left(\frac{1}{t+1}\right)} = \frac{1}{t+1}$$

$$\frac{1}{t+1}\left(x' - \frac{1}{t+1}x\right) = \left(\frac{1}{t+1}x\right)' = \frac{2}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$= \frac{A(t+2) + B(t+1)}{(t+1)(t+2)}$$

$2 = A(t+2) + B(t+1)$

$t=-1$: $2 = A(1) = A$

$t=-2$ $2 = B(-1) = -B$

$B = -2$

So $$\left(\frac{1}{t+1}x\right)' = \frac{2}{t+1} - \frac{2}{t+2}$$

So $$\frac{1}{t+1}x = \int \frac{2}{t+1} - \frac{2}{t+2}\,dt = 2\ln(t+1) - 2\ln(t+2) + C$$

So $$\boxed{x = 2(t+1)\ln|t+1| - 2(t+1)\ln|t+2| + C(t+1)}$$

2. (15 pts.) Sketch the phase diagram of the autonomous equation

$$x' = x^3 + 4x^2 + x - 6$$

Find $\lim_{t\to\infty} x(t)$ for the solutions satisfying each of the initial conditions

(a) $x(1) = -2$

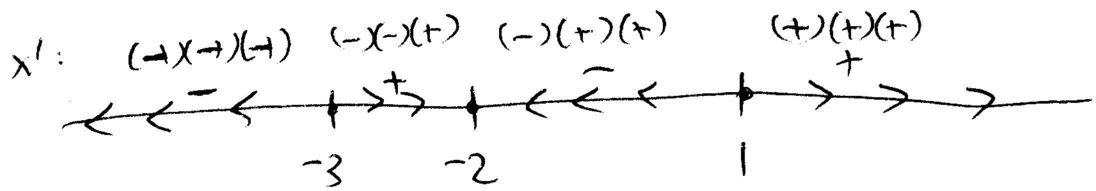
(b) $x(1) = 0$

(c) $x(1) = 2$

$x = 1$ ✓

$$x' = x^3 + 4x^2 + x - 6 = 0$$
$$= (x-1)(x^2+5x+6)$$
$$= (x-1)(x+2)(x+3)$$

$\Rightarrow x = 1, -2, -3$ equilibrium solutions



(a) $x(1) = -2$ is equilibrium, so $\lim_{t\to\infty} x(t) = -2$

(b) $x(1) = 0$ is in decreasing region $\lim_{t\to\infty} x(t) = -2$

(c) $x(1) = 2$ is in increasing region $\lim_{t\to\infty} x(t) = \infty$

3. (15 pts.) Find the solution to the initial value problem

$$x'' + x = \sec t$$
$$x(0) = 1\ ,\ x'(0) = 0$$

variation of parameters!

$x'' + x = 0$ $\qquad f(t) = \sec t$

$a^2 + 1 = 0 \qquad a = \pm i$

$\rightarrow$ fundamental solutions $x_1 = \sin t\ ,\ x_2 = \cos t$

$$W(x_1, x_2) = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = (\sin t)(-\sin t) - (\cos t)(\cos t) = -(\sin^2 t + \cos^2 t) = -1$$

particular solution:

$$x = c_1 x_1 + c_2 x_2$$

$$c_1 = \int \frac{-x_2 f}{W} dt = \int \frac{-(\cos t)(\sec t)}{-1} dt = \int \frac{\cos t}{\cos t} dt = \int 1\, dt = t$$

$$c_2 = \int \frac{x_1 f}{W} = \int \frac{(\sin t)(\sec t)}{-1} dt = -\int \frac{\sin t}{\cos t} dt$$

$u = \cos t\ ,\ du = -\sin t\, dt$ $\qquad = \int \frac{du}{u} \Big|_{u=\cos t} = \ln u \Big|_{u=\cos t} = \ln(\cos t)$

so gen'l solution is

$$x = c_1 \sin t + c_2 \cos t + t \sin t + \cos t \ln(\cos t)$$

$1 = x(0) = c_1(0) + c_2(1) + 0 + 1 \ln 1 = c_2$

$x'(t) = c_1 \cos t - c_2 \sin t + \cancel{\cos t} + t\cos t - \sin t \ln(\cos t) + \cos t \left(\frac{-\sin t}{\cos t}\right)$

$0 = x'(0) = c_1 - 0 + 0 + 0 - 0 - 0 = c_1$

$$\boxed{x(t) = \cos t + t \sin t + \cos t \ln(\cos t)}$$

4. (20 pts.) Two armies, A and B, meet on the battlefield, and according to Lanchester's Law of Combat, their combat losses are governed by the system of equations (where t = hours)

$$\cancel{A' = -\tfrac{4}{3}B} \qquad \cancel{B' = -3A}$$

If the initial sizes of the armies are $A(0) = 1000$ and $B(0) = 700$, determine each army's strength at time t. By sketching the phase plane for the system of equations (in the first quadrant $A \geq 0, B \geq 0$), determine who will lose the battle (i.e., be taken to 0 first!).

$$A' = -3B \qquad B' = -\tfrac{4}{3}A$$

$$A'' = -3B' = -3\left(-\tfrac{4}{3}A\right) = 4A \qquad A'' - 4A = 0$$

$$a^2 - 4 = 0 \qquad a = \pm 2$$

$$A = c_1 e^{2t} + c_2 e^{-2t}$$

$$B = -\tfrac{1}{3}A' = -\tfrac{1}{3}\left(c_1 2e^{2t} - c_2 2e^{-2t}\right) = -\tfrac{2}{3}c_1 e^{2t} + \tfrac{2}{3}c_2 e^{-2t}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = c_1 e^{2t}\begin{pmatrix} 1 \\ -2/3 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$$

$$A(0) = c_1 + c_2 = 1000 \qquad B(0) = -\tfrac{2}{3}c_1 + \tfrac{2}{3}c_2 = 700$$

$$\Rightarrow -c_1 + c_2 = \tfrac{3}{2}(700) = 3(350) = 1050$$

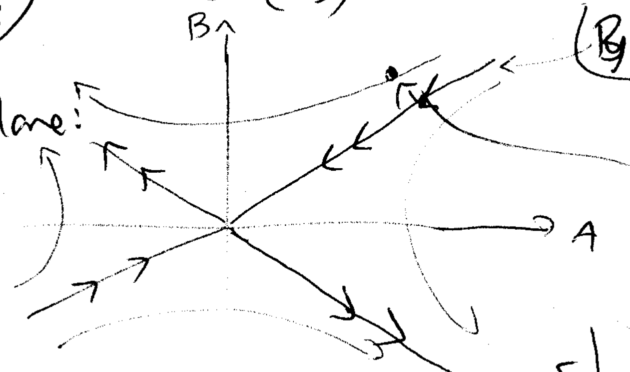
add! $2c_2 = 2050 \rightsquigarrow c_2 = 1025 \to c_1 = \cancel{c_2 - 1000} = -25$

$1000 - c_2$

$$\begin{pmatrix} A \\ B \end{pmatrix} = -25\, e^{2t}\begin{pmatrix} 1 \\ -2/3 \end{pmatrix} + 1025\, e^{-2t}\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$$

$\boxed{B = \tfrac{2}{3}A}$ $\quad 700 > \frac{2000}{3} = \frac{2}{3}(1000)$

Phase Plane!



initial value is up here

$\Rightarrow$ B stays positive &

A goes to 0

so $\boxed{\underline{A} \text{ loses.}}$

ugly!

5. (20 pts.) Use Laplace transforms to find the solution to the initial value problem

$$y'' + 4y = te^t$$
$$y(0) = 0 \ , \ y'(0) = 1$$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$

$\mathcal{L}\{y''+4y\} = \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$

$\parallel$

$s^2\mathcal{L}\{y\} - s\cdot 0 - 1 + 4\mathcal{L}\{y\}$ $\qquad (s^2+4)\mathcal{L}\{y\} = 1 + \frac{1}{(s-1)^2}$

$\mathcal{L}\{y\} = \frac{1}{s^2+4} + \frac{1}{(s-1)^2(s^2+4)}$

So $y = \mathcal{L}^{-1}\{\frac{1}{s^2+4}\} + \mathcal{L}^{-1}\{\frac{1}{(s-1)^2(s^2+4)}\}$

$\frac{1}{(s-1)^2(s^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+4} = \frac{A(s-1)(s^2+4) + B(s^2+4) + (Cs+D)(s-1)^2}{(s-1)^2(s^2+4)}$

So need $1 = A(s-1)(s^2+4) + B(s^2+4) + (Cs+D)(s-1)^2$

$s=1: \quad 1 = 0 + 5B + 0 \longrightarrow B = \frac{1}{5}$

$s=2: \quad 1 = 8A + \frac{8}{5} + 2C + D$ $\longrightarrow$ subtract! $0 = 12A + \frac{4}{5} + 2C$

$s=0: \quad 1 = -4A + \frac{4}{5} + D$ $\longrightarrow D - 4A = \frac{1}{5}$ $\quad D = 4A + \frac{1}{5}$

$s=-1: \quad 1 = -10A + 1 + 4(D-C)$ $\quad 10A = 4(D-C)$ $\quad A = \frac{2}{5}D - \frac{2}{5}C$

$A = \frac{2}{5}(4A + \frac{1}{5}) - \frac{2}{5}C$

subtract! $\quad -\frac{4}{5} = 12A + 2C$

$\frac{2}{25} = -\frac{3}{5}A + \frac{2}{5}C \longrightarrow \frac{2}{5} = -3A + 2C$

$-\frac{6}{5} = 15A \longrightarrow A = \frac{-6}{75} = \frac{-2}{25}$

$0 = -\frac{24}{25} + \frac{4}{5} + 2C$

$2C = \frac{4}{25}$ $\quad C = \frac{2}{25}$

$D = 4(\frac{-2}{25}) + \frac{1}{5} = \frac{-3}{25}$

$y = \mathcal{L}^{-1}\{\frac{1}{s^2+4}\} + \mathcal{L}^{-1}\{\frac{-2}{25}\frac{1}{s-1}\} + \mathcal{L}^{-1}\{\frac{1}{5}\frac{1}{(s-1)^2}\} + \mathcal{L}^{-1}\{\frac{2}{25}\frac{s}{s^2+4}\} + \mathcal{L}^{-1}\{\frac{-3}{25}\frac{1}{s^2+4}\}$

$= \boxed{\frac{1}{2}\sin(2t) - \frac{2}{25}e^t + \frac{1}{5}te^t + \frac{2}{25}\cos(2t) - \frac{3}{25}\sin(2t)}$

6. (15 pts) Apply Laplace transforms to the two sides of the equation

$$ty'' + y' + ty = \cos(3t)$$
$$\text{(with initial conditions } y(0) = 1\text{ , } y'(0) = 2)$$

to find an equation that the Laplace transform $\mathcal{L}\{y\} = F(s)$ must satisfy.

$$\mathcal{L}\{ty'' + y' + ty\} = \mathcal{L}\{\cos(3t)\} = \frac{s}{s^2+3^2}$$

$$\shortparallel$$

$$\mathcal{L}\{ty''\} + \mathcal{L}\{y'\} + \mathcal{L}\{ty\}$$

$$\shortparallel$$

$$-\frac{d}{ds}\mathcal{L}\{y''\} + (s\mathcal{L}\{y\} - y(0)) + \left(-\frac{d}{ds}\mathcal{L}\{y\}\right)$$

$$= -\frac{d}{ds}\left(s^2F(s) - s(1) - 2\right) + (sF(s) - 1) - \frac{d}{ds}(F(s))$$

$$= -\left(2sF(s) + s^2F'(s) - 1\right) + (sF(s) - 1) - F'(s)$$

$$= -(s^2+1)F'(s) + (s - 2s)F(s) + 1 - 1$$

$$= -(s^2+1)F'(s) - sF(s)$$

$$\text{So } \boxed{(s^2+1)F'(s) + sF(s) = \frac{-s}{s^2+9}}$$