Solution

Name:

Math 221, Section 3

Quiz number 6

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. Find the solution to the initial value problem

$$2y'' - 7y' + 3y = 0$$

$$y(0) = 1 \quad , \quad y'(0) = -7$$

Auxiliary eqn: $2r^2 - 7r + 3 = 0$

$$r = \frac{7 \pm \sqrt{49 - 4 \cdot 6}}{4} = \frac{7 \pm \sqrt{25}}{4} = \frac{7 \pm 5}{4}$$

$$= \frac{1}{2}, 3$$

Fundamental set of solutions: $y_1 = e^{\frac{1}{2}x}$, $y_2 = e^{3x}$

$$y = C_1 e^{\frac{1}{2}x} + C_2 e^{3x}$$

$$y' = \frac{1}{2} C_1 e^{\frac{1}{2}x} + 3C_2 e^{3x}$$

$y(0) = 1 = C_1 + C_2$

$y'(0) = -7 = \frac{1}{2}C_1 + 3C_2$

$C_1 + C_2 = 1$

$\frac{1}{2}C_1 + 3C_2 = -7 \qquad C_1 + 6C_2 = -14$

$5C_2 = -15 \qquad C_2 = -3$

$C_1 = 1 - C_2 = 1 - (-3) = 4$

$$\boxed{y = 4e^{\frac{1}{2}x} - 3e^{3x}}$$

2. (15 pts.) One solution to the differential equation

$$y'' - \frac{1}{t}y' + \frac{1-t}{t}y = 0$$

is $y_1(t) = e^t$.

Use reduction of order to find a second, independent, solution to the equation.

$p(t) = -\frac{1}{t}$ $\quad$ $y_2 = c(t)y_1(t)$ $\quad$ $\int p(t) = -\ln t$

$$c(t) = \int \frac{e^{-\int p(t)dt}}{(y_1(t))^2}dt = \int \frac{e^{\ln t}}{(e^t)^2}dt$$

$$= \int te^{-2t}dt \underset{\substack{u=t \quad dv=e^{-2t}dt \\ du=dt \quad v=-\frac{1}{2}e^{-2t}}}{=} -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}dt$$

$$= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}$$

So $y_2(t) = \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right)e^t = \boxed{-\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}}$

Check!

$$y_2' = -\frac{1}{2}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} = \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}$$

$$y_2'' = \frac{1}{2}e^{-t} - \frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} = \frac{3}{4}e^{-t} - \frac{1}{2}te^{-t}$$

$$y_2'' - \frac{1}{t}y_2' + \left(\frac{1}{t}-1\right)y = \left(\frac{3}{4}e^{-t} - \frac{1}{2}te^{-t}\right) - \left(\frac{1}{2}e^{-t} - \frac{1}{4t}e^{-t}\right)$$
$$+\left[\left(-\frac{1}{2}e^{-t} - \frac{1}{4t}e^{-t}\right) - \left(-\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}\right)\right]$$

$$= \left(\frac{3}{4} - \frac{1}{2} - \frac{1}{2} + \frac{1}{4}\right)e^{-t} + \left(-\frac{1}{2} + \frac{1}{2}\right)te^{-t} + \left(\frac{1}{4} - \frac{1}{4}\right)\frac{1}{t}e^{-t} = 0 + 0 + 0$$
$$= 0 \checkmark!$$

3. (20 pts.) The functions

$$y_1(t) = t \text{ and } y_2(t) = te^t$$

form a fundamental set of solutions to the differential equation

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0\ .$$

Use variation of parameters to find a particular solution to the equation

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = t^3\ . = g(t)$$

$y = c_1y_1 + c_2y_2$ where $c_1'y_1 + c_2'y_2 = 0$

$c_1'y_1' + c_2'y_2' = g(t)$

$c_1 = \int \frac{-gy_2}{W}dt \qquad c_2 = \int \frac{gy_1}{W}dt \qquad W = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix}$

$= \int \frac{-t^3 te^t}{t^2e^t}dt \qquad = \int \frac{t^3 \cdot t}{t^2 e^t}dt \qquad = t^2e^t + te^t - te^t$

$= t^2e^t$

$= \int -t^2\,dt = -\frac{t^3}{3} \qquad = \int t^2e^{-t}dt$

$u = t^2 \quad dv = e^{-t}dt$

$du = 2t\,dt \quad v = -e^{-t}$

$= -t^2e^{-t} + 2\int te^{-t}dt$

$u = t \quad dv = e^{-t}dt$

$du = dt \quad v = -e^{-t}$

$= -t^2e^{-t} + 2(-te^{-t} + \int e^{-t}dt)$

$= -t^2e^{-t} - 2te^{-t} - 2e^{-t}$

$$\boxed{y = \left(-\frac{t^3}{3}\right)(t) + (-t^2e^{-t} - 2te^{-t} - 2e^{-t})(te^t)}$$

$= -\frac{1}{4}t^4 - t^3 - 2t^2 - 2t$

4. (15 pts.) Find the general solution to the differential equation

$$y''' - 2y'' + 2y' - y = 2e^t - \sin t$$

$r=1$?

$1-2+2-1=0$ ✓

$y''' - 2y'' + 2y' - y = 0$

$r^3 - 2r^2 + 2r - 1 = 0$

$(r-1)(r^2-r+1) = 0$

$r=1, \quad r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$y_1 = e^t, \quad y_2 = e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t, \quad y_3 = e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$

$y''' - 2y'' + 2y' - y = 2e^t$

guess ~~$y = Ae^t, y' = Ae^t$~~ ~~$y'' = Ae^t, y''' = Ae^t$~~

$y = Ate^t, \quad y' = Ate^t + Ae^t$

$y'' = Ate^t + 2Ae^t \quad y''' = Ate^t + 3Ae^t$

~~$Ate^t - 2Ate^t - 2Ae^t$~~

$\cancel{Ate^t} + 3Ae^t - \cancel{2Ate^t} - 4Ae^t + \cancel{2Ate^t} + 2Ae^t - \cancel{Ate^t}$

$= (3A - 4A + 2A)e^t = Ae^t = 2e^t$

$A = 2$

$y''' - 2y'' + 2y' - y = -\sin t$

guess

$y = A\sin t + B\cos t$

$y' = A\cos t - B\sin t$

$y'' = -A\sin t - B\cos t$

$y''' = -A\cos t + B\sin t$

$y''' - 2y'' + 2y' - y = -1 \cdot \sin t + 0 \cdot \cos t$

$(B + 2A - 2B - A)\sin t + (-A + 2B + 2A - B)\cos t$

$= (A - B)\sin t + (A + B)\cos t$

$A - B = -1 \quad A + B = 0 \quad B = -A$

$2A = -1 \quad A = -\frac{1}{2}, \ B = \frac{1}{2}$

$$\boxed{y = c_1 e^t + c_2 e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + c_3 e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t + 2te^t + -\frac{1}{2}\sin t + \frac{1}{2}\cos t}$$

5. (15 pts.) Find the solution to the initial value problem

$$x' = 2x - 9y \qquad x(0) = 1$$
$$y' = x + 2y \qquad y(0) = -1$$

$$\cancel{x(0)=1 \quad x'(0)=2(1)-9(-1)=11}$$
$$y(0)=-1 \quad y'(0)=(1)+2(-1)=-1$$

$$x = y' - 2y$$

$$y'' - 2y' = 2(y'-2y) - 9y$$
$$y'' - 2y' = 2y' - 13y$$
$$y'' - 4y' + 13y = 0$$

$$r^2 - 4r + 13 = 0 \qquad r = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$$
$$= 2 \pm 3i$$

$$y(t) = \cancel{e}\, c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)$$

$$y(0) = -1 = c_1 \cdot 1 \cdot 1 + c_2 \cdot 1 \cdot 0 = c_1 \qquad c_1 = -1$$

$$y'(t) = 2c_1 e^{2t}\cos(3t) - 3c_1 e^{2t}\sin(3t) + 2c_2 e^{2t}\sin(3t) + 3c_2 e^{2t}\cos(3t)$$

$$-1 = y'(0) = 2c_1 - 0 + 0 + 3c_2$$
$$3c_2 = -1 - 2c_1 = -1 + 2 = 1 \qquad c_2 = \frac{1}{3}$$

$$\boxed{y = -e^{2t}\cos(3t) + \frac{1}{3}e^{2t}\sin(3t)}$$

$$x = y' - 2y = \left(-2e^{2t}\cos(3t) + 3e^{2t}\sin(3t) + \frac{2}{3}e^{2t}\sin(3t) + e^{2t}\cos(3t)\right) - 2\left(-e^{2t}\cos(3t) + \frac{1}{3}e^{2t}\sin(3t)\right)$$
$$= (-2+1+2)e^{2t}\cos(3t) + \left(3 + \frac{2}{3} - \frac{2}{3}\right)e^{2t}\sin(3t)$$
$$= e^{2t}\cos(3t) + 3e^{2t}\sin(3t)$$

6. (20 pts.) Sketch the direction field for the system of equations

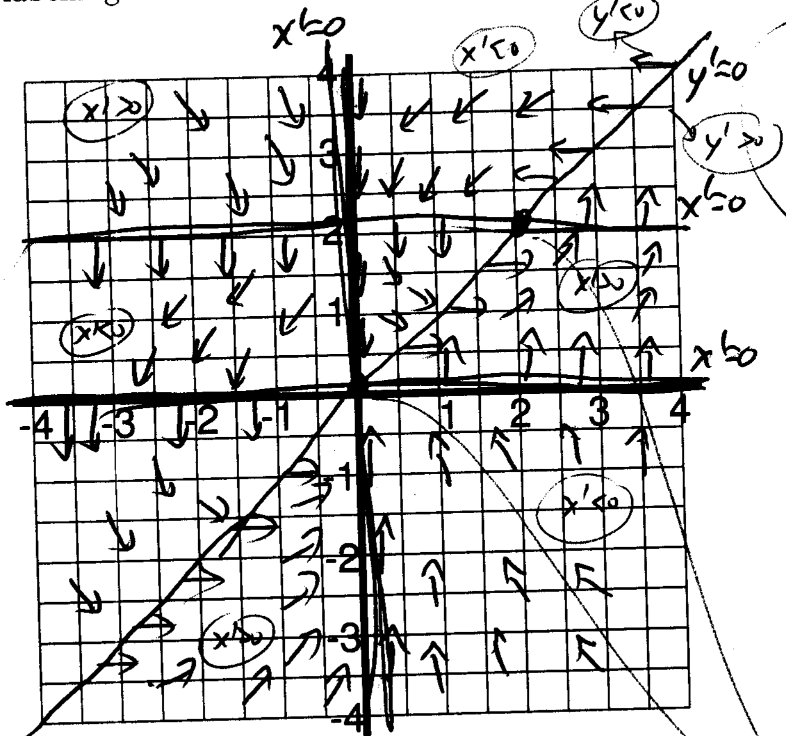
$$x' = 2xy - xy^2$$
$$y' = x - y$$

labelling all nullclines and equilibrium solutions.

$x' = 2xy - xy^2 = 0$

$= xy(2-y)$

$x=0$
$y=0$
$y=2$
— vertical



$y' = x-y = 0$

$\boxed{y = x}$ horizontal

equilibrium points