Topos Internal Language

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Subobject Classifiers

Let $\mathcal{C}$ be a category with terminal object 1. A *subobject classifier* is an object Ω and a morphism $\mathtt{true} : 1 \to \Omega$ such that for each monomorphism $m : A \to B$, there exists a unique morphism $\omega(m) : B \to \Omega$ such that the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "m"] & 1 \arrow[d, tail, "\mathtt{true}"] \\ B \arrow[r, dashed, "\omega(m)"] & \Omega \end{tikzcd}$$

is a pullback. Ω is called the *object of truth values*, and $\omega(m)$ is the *classifying morphism* of m.

Examples of Subobject Classifiers

In **Set**, $1 = \{*\}$, $\Omega = \{t, f\}$, and $\texttt{true}(*) = t$.

Let $i : A \to B$ be an injective function. Then the classifying map $\omega(i) : B \to \Omega$ is given by

$$\omega(i)(b) = \begin{cases} t & b \in i(A) \\ f & b \notin i(A) \end{cases}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & \{*\} \arrow[d, tail, "\texttt{true}"] \\ B \arrow[r, dashed, "\omega(i)"] & \{t, f\} \end{tikzcd}$$

Examples of Subobject Classifiers

Denote by $\mathbf{Sh}(X)$ the category of sheaves of sets on a fixed topological space X, whose arrows are morphisms of sheaves.

- $\mathbf{Sh}(X)$ has a terminal object 1 which sends each open $U \subseteq X$ to the singleton $\{*\}$;
- $\mathbf{Sh}(X)$ has object of truth values Ω satisfying $\Omega(U) = \mathrm{Open}_X(U)$;
- The subobject classifier $\mathtt{true} : 1 \to \Omega$ satisfies $\mathtt{true}(U)(*) = U$ for all U open in X.

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Given a monomorphism of sheaves $j : \mathcal{G} \to \mathcal{F}$, the classifying morphism $\omega(j) : \mathcal{F} \to \Omega$ is given by the family of maps $\omega(j)(U) : \mathcal{F}(U) \to \Omega(U)$ where

$$\omega(j)(U)(s) = \bigcup_{\rho^U_V(s) \in j(V)(\mathcal{G}(V))} V.$$

Classifying Subobjects

A *subobject* of X is an equivalence class of pairs of an object Y with a monomorphism $\alpha : Y \to X$ under the equivalence relation $(Y, \alpha) \sim (Z, \beta)$ if $\omega(\alpha) = \omega(\beta)$.

Subobjects (Y, α) of X can be uniquely classified by their classifying morphism $\omega(\alpha) : X \to \Omega$. In fact,

$$\mathrm{Sub}_{\mathcal{C}}(X) \cong \mathrm{Hom}_{\mathcal{C}}(X, \Omega).$$

Exponentials

Let X, Y be objects in $\mathcal{C}$. An object X^Y together with a map $\texttt{eval} : X^Y \times Y \to X$ is an *exponential* if for any object Z and morphism $g : Z \times Y \to X$ there is a unique morphism $\lambda g : Z \to X^Y$ such that the diagram

$$\begin{tikzcd}
Z \times Y \arrow[d, "\lambda g \times \mathrm{id}_Y"'] \arrow[dr, "g"] & \\
X^Y \times Y \arrow[r, "\texttt{eval}"] & X
\end{tikzcd}$$

commutes. Here, λg is called the *transpose* of g.

This transposition property gives us the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\left(Z, X^Y\right) \cong \mathrm{Hom}_{\mathcal{C}}(Z \times Y, X).$$

Toposes (Topoi)

A *topos* is a category $\mathcal{E}$ satisfying the following properties:

1. $\mathcal{E}$ has a terminal object 1;
2. Any two objects X and Y have a product $X \times Y$ in $\mathcal{E}$;
3. Any two objects X and Y have an exponential X^Y in $\mathcal{E}$;
4. $\mathcal{E}$ has finite limits;
5. $\mathcal{E}$ has a subobject classifier $\mathtt{true} : 1 \to \Omega$.

A category satisfying the first three properties is said to be *Cartesian closed*.

Example: The category $\mathbf{Sh}(X)$ of sheaves over a topological space X is a topos.

Heyting Algebra

A *Heyting algebra* is a poset H with minimal element 0 and maximal element 1 and operations $\wedge$ and $\vee$ defined as

$$x \wedge y = \inf\{x, y\} \qquad x \vee y = \sup\{x, y\}$$

and an operation $\Rightarrow$ satisfying the condition

$$z \leq (x \Rightarrow y) \text{ iff } (z \wedge x) \leq y.$$

We consequently define $\neg x$ as $x \Rightarrow 0$.

Example of a Heyting Algebra

Let X be a topological space. Then $\mathrm{Open}_X(X)$ is a Heyting algebra with

- $1 = X$;
- $0 = \varnothing$;
- $U \leq V$ means $U \subseteq V$;
- $U \wedge V = U \cap V$;
- $U \vee V = U \cup V$;
- $(U \Rightarrow V) = \mathrm{int}(V \cup (X \setminus U))$.

In this case, $\neg U = \mathrm{int}(X \setminus U)$. (Note that $\neg\neg U \neq U$ in general.)

Heyting Algebra of $\mathrm{Sub}_{\mathcal{E}}(1)$

Let $\mathcal{E}$ be a topos. Recall $\mathrm{Sub}_{\mathcal{E}}(1) \cong \mathrm{Hom}_{\mathcal{E}}(1, \Omega)$ is the collection of subobjects of 1. Given an object X whose unique map $\alpha_X : X \to 1$ is a monomorphism, we have $\omega(\alpha_X) : 1 \to \Omega$ is in $\mathrm{Sub}_{\mathcal{E}}(1)$. We can realize this collection as a Heyting algebra in the following sense:

- `true` is the maximal object;
- `false` $= \omega(\alpha_0)$ is the minimal object (0 is the initial object);
- $X \leq Y$ means there is a monomorphism from X to Y;
- $X \wedge Y = X \times Y$;
- $X \vee Y = X \amalg Y$;
- $(X \Rightarrow Y) = Y^X$.

In this case, $\neg X = \omega(\alpha_{0^X})$.

Note by the transposition property, we fulfill the required condition

$$Z \subseteq Y^X \text{ iff } Z \times X \subseteq Y.$$

The (very) Basics of Type Theory

A *type system* is a formal system in which every term has a "type" which defines its meaning.

Ordinary mathematical statements and theorems can be formulated in the symbolism of standard logic. This symbolism starts with constants and variables (0, 1, x, ...). For example, $x \in \mathbb{N}$ means x is a variable of *type* $\mathbb{N}$.

These symbols combine with operations to give *terms* (x^2, $x + y$, ...) of a given type. These terms can yield *formulas* ($x < y$, $x + y = z$, ...).

Formulas can combine with propositional connectives ($\wedge$, $\vee$, $\Rightarrow$, ...) and quantifiers ($\forall$, $\exists$, ...) to form more formulas.

Mitchell-Bénabou Language

The Mitchell-Bénabou Language of a topos $\mathcal{E}$ is a formal language wherein:

- the *types* are the objects of $\mathcal{E}$;
- the *variables* of type A are interpreted as identity morphisms $\mathrm{id}_A : A \to A$ in $\mathcal{E}$;
- the *terms* of type B in variables x_i of type X_i are interpreted as morphisms from the product of the X_i to B;
- the *formulas* are terms of type Ω.

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- the *formulas* are terms of type Ω.

The rules of inference appropriate to a general topos follow the structure of the Heyting algebra and are precisely the standard rules for *intuitionism*. Mac Lane and Moerdijk call this a “striking observation.”

Intuitionism

Intuitionism is a logic system in which we forgo:

- the axiom of choice,
- the law of the excluded middle ($x \vee \neg x = 1$),
- and the law of double negation ($\neg\neg x = x$).

Without these laws, we cannot use contradiction to prove something is true.
We also can only prove existence by construction: this is also called *constructive logic*.

Example: In intuitionistic logic, one can prove that any *inhabited* subset of $\mathbb{N}$ does *not not* possess a minimal element. Moreover, every *detachable* inhabited subset of $\mathbb{N}$ possesses a minimal element.

Topos Internal Language

The Mitchell-Bénabou Language allows us to "pretend" that objects in a topos have variables that we can interact with. Then we can use familiar proofs to conclude useful things about the objects of the topos (as long as such proofs are valid in the internal language).

Blechschmidt: "Any (intuitionistically valid) theorem about modules yields a corresponding theorem about sheaves of modules."

Kripke-Joyal Semantics

Let φ be a formula and let U be an open set. The meaning of

$$U \vDash \varphi$$

is "φ holds on U."

We also often write $s : \mathcal{F}$ as opposed to $s \in \mathcal{F}$ to denote variables s of type $\mathcal{F}$.

The Kripke-Joyal Dictionary

$$
\begin{aligned}
U \vDash s = t : \mathcal{F} &:\Longleftrightarrow s|_U = t|_U \in \mathcal{F}(U). \\
U \vDash \bigwedge_{j \in J} \varphi_j &:\Longleftrightarrow \text{for all } j \in J,\ U \vDash \varphi_j. \\
U \vDash \bigvee_{j \in J} \varphi_j &:\Longleftrightarrow \text{there is a covering } U = \bigcup_{i \in I} U_i \text{ such that for all } i, \\
&\qquad U_i \vDash \varphi_j \text{ for some } j \in J. \\
U \vDash \varphi \Rightarrow \psi &:\Longleftrightarrow \text{for all open } V \subseteq U,\ V \vDash \varphi \text{ implies } V \vDash \psi. \\
U \vDash \forall s : \mathcal{F}.\varphi(s) &:\Longleftrightarrow \text{for all } s \in \mathcal{F}(V) \text{ on open } V \subseteq U,\ V \vDash \varphi(s). \\
U \vDash \exists s : \mathcal{F}.\varphi(s) &:\Longleftrightarrow \text{there is a covering } U = \bigcup_{i \in I} U_i \text{ such that for all } i : \\
&\qquad \text{there is an } s_i \in \mathcal{F}(U_i) \text{ such that } U_i \vDash \varphi(s_i).
\end{aligned}
$$

Internal vs External Example

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is a monomorphism of sheaves if and only if, from the internal perspective, α is simply an injective map.

Proof (Blechschmidt).

$$
\begin{aligned}
& X \models \ulcorner \alpha \text{ is injective.} \urcorner \\
\iff & X \models \forall s : \mathcal{F}. \forall t : \mathcal{F}. \alpha(s) = \alpha(t) \Rightarrow s = t \\
\iff & \text{for all open } U \subseteq X, \text{ sections } s \in \mathcal{F}(U) : \\
& \text{for all open } V \subseteq U, \text{ sections } t \in \mathcal{F}(V) : V \models \alpha(s) = \alpha(t) \Rightarrow s = t \\
\iff & \text{for all open } W \subseteq V \subseteq U \subseteq X \text{ with } s \in \mathcal{F}(U), t \in \mathcal{F}(V) \\
& \alpha(W)(\rho^U_W(s)) = \alpha(W)(\rho^V_W(t)) \Rightarrow \rho^U_W(s) = \rho^V_W(t) \\
\iff & \text{for all open } U \subseteq X,\ s, t \in \mathcal{F}(U) : \alpha(U)(s) = \alpha(U)(t) \Rightarrow s = t \\
\iff & \alpha \text{ is a monomorphism of sheaves.}
\end{aligned}
$$

$\square$

Finite Type Sheaves

For a locally ringed space $(X, \mathcal{O}_X)$, recall that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is of finite type if there is an open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ and natural numbers n_λ such that

$$(\mathcal{O}_X|_{U_\lambda})^{n_\lambda} \longrightarrow \mathcal{F}|_{U_\lambda}$$

is a surjective morphism of sheaves.

From the internal perspective, $\mathcal{F}$ is of finite type if it, considered as an ordinary module, is finitely generated. That is,

$$X \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}. \forall x : \mathcal{F}. \exists a_1, \ldots, a_n : \mathcal{O}_X . x = \sum_i a_i x_i.$$

Short Exact Sequence of Finite-Type Sheaves

Theorem: Let $(X, \mathcal{O}_X)$ be a locally ringed space and let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{H}$ are of finite type, then $\mathcal{G}$ has finite type.

External Method: But First a Lemma!

Lemma: A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules has finite type if and only if for all $x \in X$, the stalk $\mathcal{F}_x$ is a finitely-generated $\mathcal{O}_{X,x}$-module and $\mathrm{rank}(\mathcal{F}_x)$ has a local upper bound: for all $x \in X$ there is a neighborhood U_x of x and a natural number n_x such that for all $y \in U_x$, $\mathrm{rank}(\mathcal{F}_y) \leq n_x$.

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Proof of Lemma.

($\Rightarrow$) We know there is a open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X and naturals n_λ such that $(\mathcal{O}_X|_{U_\lambda})^{n_\lambda} \to \mathcal{F}|_{U_\lambda}$ is a surjective morphism of sheaves. Then for all $x \in U_\lambda$, $\mathcal{O}_{X,x}^{n_\lambda} \to \mathcal{F}_x$ is a surjection of $\mathcal{O}_{X,x}$-modules. Thus $\mathrm{rank}(\mathcal{F}_x) \leq n_\lambda$. This is an upper bound for all $x \in U_\lambda$.

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Lemma: A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules has finite type if and only if for all $x \in X$, the stalk $\mathcal{F}_x$ is a finitely-generated $\mathcal{O}_{X,x}$-module and $\operatorname{rank}(\mathcal{F}_x)$ has a local upper bound: for all $x \in X$ there is a neighborhood U_x of x and a natural number n_x such that for all $y \in U_x$, $\operatorname{rank}(\mathcal{F}_y) \leq n_x$.

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($\Leftarrow$) Suppose the rank of each stalk has a local upper bound, as defined above. Then the $\{U_x\}_{x \in X}$ form an open cover of X wherein $(\mathcal{O}_X|_{U_x})^{n_x} \to \mathcal{F}|_{U_x}$ is a surjective morphism of sheaves. Thus $\mathcal{F}$ is of finite type. $\square$

External Method

Proof of Theorem.

We know that on each stalk

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Then we know that $\text{rank}(\mathcal{F}_x) + \text{rank}(\mathcal{H}_x) = \text{rank}(\mathcal{G}_x)$ from commutative algebra.

Since $\mathcal{F}$ and $\mathcal{H}$ are finite type, the ranks of their stalks have local upper bounds by the lemma. Thus the rank of $\mathcal{G}_x$ has a local upper bound for each $x \in X$. By the lemma, $\mathcal{G}$ is then of finite type. $\square$

Internal Method

Proof (Blechschmidt).

From the internal perspective, we are given a short exact sequence of modules with the outer two finitely generated and we have to show that the middle one is finitely generated. It is well-known that this follows; and since the usual proof of this fact is intuitionistically valid, we are done.

Internal Method

Proof (Blechschmidt).

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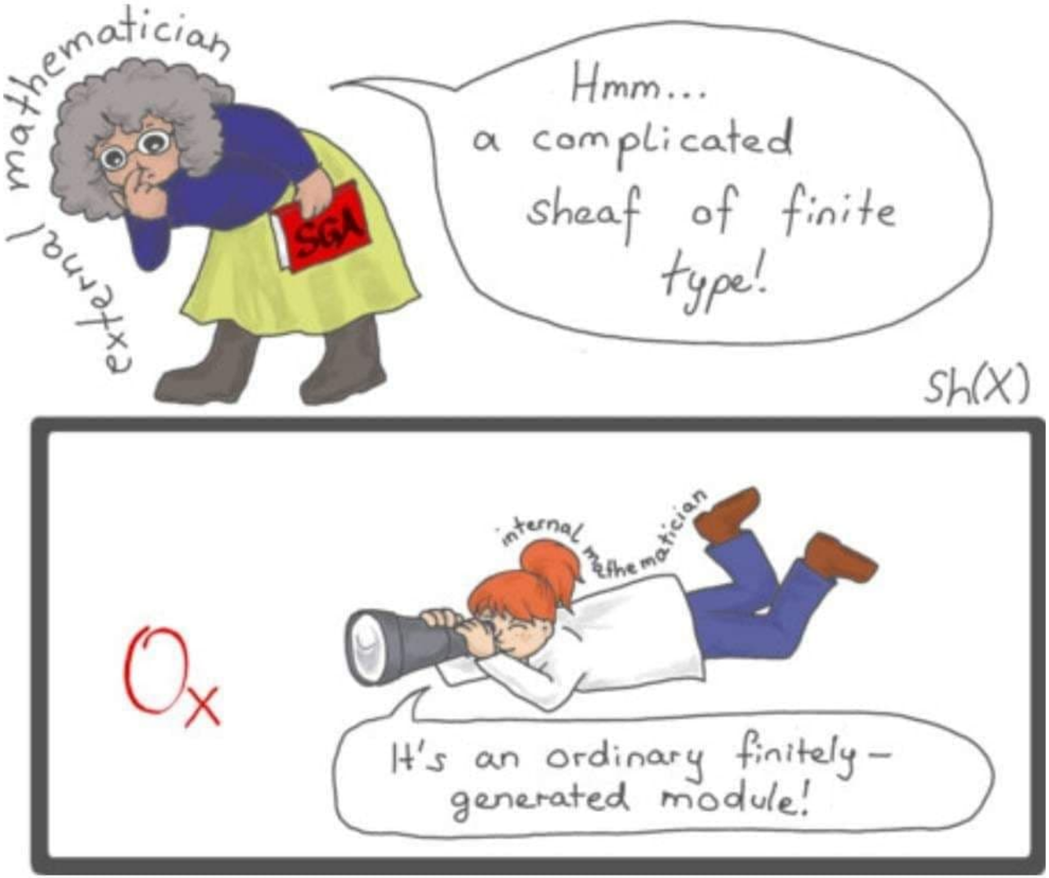
Picture


external mathematician
Hmm...
a complicated
sheaf of finite
type!
Sh(X)
SGA
internal mathematician
$\mathcal{O}_X$
It's an ordinary finitely-
generated module!

References

Ingo Blechschmidt's *Using the internal language of toposes in algebraic geometry* (2017).

Saunders Mac Lane and Ieke Moerdijk's *Sheaves in Geometry and Logic* (1992).