ACYCLICITY OF TATE CONSTRUCTIONS

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Abstract. We prove that a Tate construction $A(u_1, \ldots, u_n \mid \partial(u_i) = z_i)$ over a DG algebra $A$, on cycles $z_1, \ldots, z_n$ in $A_{\geq 1}$, is acyclic if and only if the map of graded-commutative algebras $H_0(A)[y_1, \ldots, y_n] \rightarrow H(A)$, with $y_i \mapsto \text{cls}(z_i)$, is an isomorphism. This is used to establish that if a large homomorphism $R \rightarrow S$ has an acyclic closure $R(U)$ with $\sup \{ i \mid U_i \neq \emptyset \} = s < \infty$, then $s$ is either 1 or an even integer.

Let $A$ be a Differential Graded algebra (henceforth abbreviated to DG algebra), and let $w_1, \ldots, w_n$ be classes in $H(A)$. Choose cycles $z_1, \ldots, z_n$ in $A$ with $\text{cls}(z_i) = w_i$ for each $i$, and consider the DG algebra $A(U) = A(u_1, \ldots, u_n \mid \partial(u_i) = z_i)$; the Tate construction on $A$ over the cycles $w_1, \ldots, w_n$, cf. [1, (6.1)].

Our main result, Theorem 2.3, asserts when $|w_i| \geq 1$ for each $i$, the Tate construction $A(U)$ is acyclic if and only if the canonical map of graded-commutative algebras:

$$H_0(A)[y_1, \ldots, y_n] \rightarrow H(A) \quad \text{where} \quad y_i \mapsto w_i,$$

is an isomorphism; here $H_0(A)[y_1, \ldots, y_n]$ denotes the free graded-commutative algebra over $H_0(A)$ on the graded set $\{y_1, \ldots, y_n\}$, cf. 1.1. Our theorem generalizes a result of Blanco, Majadas, and Rodicio [4, Theorem 1] who consider the case where $A$ is the Koszul complex on a set of elements in a ring and each class $w_i$ has degree one.

The crux of the matter is the interplay between regularity, in the sense of Tate [10], of a sequence of elements in a graded-commutative algebra and quasi-regularity, a concept introduced here by extrapolating from commutative rings, which are viewed as graded-commutative algebras concentrated in degree 0. In this context, we prove that, as in the classical case, a regular sequence is quasi-regular, while the converse holds under suitable separatedness assumptions. This is the content of Section 1.

Section 2 combines the tools developed in Section 1 with those of Tate, to arrive at the following: Let $A(U)$ be the Tate construction over $A$ on a classes $w = w_1, \ldots, w_n$ in $H(A)$; we make no assumptions on the degrees of the elements. Then, under separatedness assumptions, the following are equivalent.

(a) the sequence $w_1, \ldots, w_n$ is regular;
(b) the sequence $w_1, \ldots, w_n$ is quasi-regular;
(c) the canonical map $H(A) \rightarrow H(A(U))$ is surjective;
(d) the canonical map $H(A) \rightarrow H(A(U))$ is an isomorphism.

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This is the essence of Theorem 2.2. The equivalence of (a), (c), and (d) is well known; thus the only new input is (a) \( \iff \) (b), which is subject of Section 1. Now, when \( A \) is a commutative ring (concentrated in degree 0), \( A(U) \) is exactly the Koszul complex on the \( z_i \)'s, so we recover a fact well known in the theory of commutative rings. In the special case where \( |z_i| \geq 1 \) for all \( i \), we build on the equivalence above to arrive at Theorem 2.3 announced at the beginning of this introduction.

From our point of view these results are handy tools to possess when studying DG algebras. One justification for this claim is that they may be used to prove the following theorem:

Let \( R \to S \) be a large homomorphism of local rings and \( R(U) \) the acyclic closure of \( S \) over \( R \). If \( s = \{ i \mid U_i \neq \emptyset \} < \infty \), then \( s \) is either 1 or an even integer.

Section 3 is dedicated to the proof of this result. The reader will find large homomorphisms defined in loc. cit.; for the moment it suffices to note that for a local ring \( R \) with residue field \( k \), the canonical surjection \( R \to k \) is large, as is any surjective local homomorphism \( R \to S \) which admits a section. In fact, in these cases one has a stronger conclusion: \( s \leq 2 \). This is well known for the map \( R \to k \), cf. [1, (7.3)] and the references therein, and for split homomorphisms it follows from work of Avramov and the author [3]. Since these two cases do not exhaust the class of large homomorphisms it may be worthwhile to report the result above. Now, the special cases mentioned do raise the question: Is \( s \leq 2 \) in general?

When the characteristic of the residue field of \( R \) is 0, this question is equivalent to a conjecture of Quillen [9, (5.6)] on the vanishing of cotangent homology, for the particular case of a large homomorphism.

1. Regularity vs. Quasi-regularity

In this note, it is expedient to adopt the convention that all rings and modules are defined over an ubiquitous commutative ring \( k \). For an element \( x \) in a graded set \( X = \{ X_i \}_{i \geq 0} \), we write \( |x| = i \) to indicate that \( x \in X_i \); this is the degree of \( x \).

An algebra \( R = \{ R_i \}_{i \geq 0} \) consists of \( k \)-modules \( R_i \) with a product compatible with the degrees and satisfying the usual properties.

A graded-commutative algebra is an algebra \( R = \{ R_i \}_{i \geq 0} \) such that

\[
r \cdot s = (-1)^{|r||s|} s \cdot r \quad \text{for} \quad r, s \in R \quad \text{and} \quad r^2 = 0 \quad \text{when} \quad |r| \text{ is odd}.
\]

A commutative ring, in the usual sense of the word, will be viewed as a graded-commutative algebra over \( \mathbb{Z} \), concentrated in degree 0.

1.1. Let \( R \) be a graded-commutative algebra. The free graded-commutative algebra over \( R \) on a set of variables \( X = \{ X_i \}_{i \geq 0} \) is the tensor product over \( R \) of the polynomial algebra on \( X_{\text{even}} \) and the exterior algebra on \( X_{\text{odd}} \); it is denoted \( R[X] \).

It is characterized by the following universal property:

If \( \phi: R \to S \) is a morphism of graded-commutative algebras and \( \alpha: X \to S \) a degree 0 map of graded sets, then there is a unique morphism of graded-commutative algebras \( \bar{\alpha}: R[X] \to S \) extending \( \phi \) and such that \( \bar{\phi}(x) = \alpha(x) \) for each \( x \in X \).

An element in \( R[X] \) is called a polynomial, although it may involve both polynomial and exterior variables. It is useful to keep in mind that \( R[X] \) has a second grading, the external grading denoted \( \deg(\cdot) \), with \( \deg(r) = 0 \) for \( r \in R \) and \( \deg(x) = 1 \) for each \( x \in X \). In general, \( R[X] \) is not a graded-commutative algebra under the external grading.
1.2. An element \( r \in R \) is regular if \( r \) is not invertible and

\[
(0: R \, r) = \begin{cases} 
0 & \text{if } |r| \text{ is even} \\
(r) & \text{if } |r| \text{ is odd}
\end{cases}
\]

Thus, a regular element has the smallest possible annihilator. A sequence \( r_1, \ldots, r_n \) in \( R \) is a regular sequence if \( r_i+1 \) is regular on \( R/(r_1, \ldots, r_i) \) for \( 0 \leq i \leq n - 1 \).

For example, for \( X = \{x_1, \ldots, x_n\} \) a set of variables over \( R \), the sequence \( x_1, \ldots, x_n \) is regular on \( R[X] \).

1.3. Let \( R \) be a graded-commutative algebra and \( I \) an ideal in \( R \). Consider the bi-graded \( k \)-module \( \text{gr}_I(R) \) with

\[
\text{gr}_I(R)_{i,n} = \left( \frac{I^n}{I^{n+1}} \right)_i 
\]

the index \( i \) is the internal degree, denoted \( |\cdot| \), and \( n \) is the external degree, denoted \( \text{deg}(\cdot) \). The product on \( R \) induces one on \( \text{gr}_I(R) \) which is compatible with both the degrees: \( |ab| = |a| + |b| \) and \( \text{deg}(ab) = \text{deg}(a) + \text{deg}(b) \) for \( a, b \in \text{gr}_I(R) \).

The commutativity of \( R \) ensures that for \( a, b \in \text{gr}_I(R) \), one has \( ab = (-1)^{|a||b|}ba \), so that \( \text{gr}_I(R) \), when graded by the internal degree alone, is a graded-commutative algebra. This is the associated graded algebra of \( R \) with respect to \( I \). Note that the canonical inclusion \( R/I = (\text{gr}_I(R))_{*,0} \hookrightarrow \text{gr}_I(R) \) is a morphism of graded-commutative algebras.

1.4. Let \( r = r_1, \ldots, r_n \) be elements in a graded-commutative algebra \( R \), let \( I = (r) \), and let \( x_1, \ldots, x_n \) a set of variables with \( |x_i| = |r_i| \) for each \( i \). By the universal property of the free graded-commutative algebra, cf. 1.1, the canonical inclusion \( R/I \hookrightarrow \text{gr}_I(R) \) extends to a map of graded-commutative algebras

\[
\varphi_r: \frac{R}{I}[x_1, \ldots, x_n] \to \text{gr}_I(R) \quad \text{with} \quad \varphi_r(x_i) = r_i
\]

This map is surjective; taking a cue from the classical case, the sequence \( r_1, \ldots, r_n \) is said to be quasi-regular if the map \( \varphi_r \) is an isomorphism. The issue is only whether the map in question is injective, so in analyzing this map, we can exploit the fact that it respects the external grading as well. Observe that if \( r_1, \ldots, r_n \) is quasi-regular, then so is any permutation of the \( r_i \)’s.

Unraveling the definition leads to the following test for quasi-regularity.

1.5. Let \( r_1, \ldots, r_n \) be elements in a graded-commutative algebra \( R \), let \( I \) be the ideal generated by the \( r_i \)’s, and let \( R[X] = R[x_1, \ldots, x_n] \), with \( |x_i| = |r_i| \) for all \( i \). The following conditions are equivalent:

(i) The sequence \( r_1, \ldots, r_n \) is quasi-regular;

(ii) For a polynomial \( F \in R[X] \) with \( \text{deg}(F) = v \) (the external degree !), if \( F(r_1, \ldots, r_n) \in I^{v+1} \), then \( F \in IR[X] \);

(iii) For an \( F \in R[X] \) with \( \text{deg}(F) = v \) if \( F(r_1, \ldots, r_n) = 0 \), then \( F \in IR[X] \).

The proof is as in [8, §16].

One may verify with no difficulty that the sequence \( x_1, \ldots, x_n \) in the free graded-commutative algebra \( R[x_1, \ldots, x_n] \) is quasi-regular. The following result is a partial converse.
1.6. Proposition. Let \( R \) be a graded-commutative algebra, let \( r_1, \ldots, r_n \in R \) be a quasi-regular sequence with \( |r_i| \geq 1 \) for each \( i \), and let \( y_1, \ldots, y_n \) be variables with \( |y_i| = |r_i| \). The morphism of graded-commutative algebras \( R_0[y_1, \ldots, y_n] \to R \), with \( y_i \mapsto r_i \) is injective.

Proof. To begin with, any polynomial in \( R_0[Y] = R_0[y_1, \ldots, y_n] \) may be viewed as a polynomial in \( R[Y] \) via the canonical inclusion \( R_0[Y] \hookrightarrow R[Y] \). Moreover, \( R_0[Y] \cap IR[Y] = \{ 0 \} \), for the ideal \( I = (r_1, \ldots, r_n) \).

Suppose that \( F \in R_0[Y] \) is a non-zero polynomial in the kernel of the map \( R_0[Y] \to R \), that is to say, \( F(r_1, \ldots, r_n) = 0 \). We may write \( F = \sum_{i=a}^b F_i(r_1, \ldots, r_n) \in I^{a+1} \), so 1.5 yields \( F_a \in IR[Y] \); hence, \( F_a \in R_0[Y] \cap IR[Y] = \{ 0 \} \), a contradiction. \( \Box \)

The next two propositions assert that, as for commutative rings, quasi-regularity is closely related to regularity.

1.7. Proposition. If a sequence \( r_1, \ldots, r_n \) in a graded-commutative algebra \( R \) is regular, then it is quasi-regular.

The converse holds under appropriate separatedness assumptions. In what follows, the Jacobson radical for a commutative ring \( Q \) is denoted \( \text{Jac}(Q) \).

1.8. Proposition. Let \( r_1, \ldots, r_n \) be elements in a graded-commutative algebra \( R \) such that either \( |r_i| \geq 1 \) for all \( i \), or each \( R_j \) is noetherian over \( R_0 \) and if \( |r_i| = 0 \), then \( r_i \) is contained \( \text{Jac}(R_0) \).

If the sequence \( r_1, \ldots, r_n \) is quasi-regular, then it is regular.

The proofs of these results are similar to those for commutative rings as found in Matsumura [8, §16], although twice as long, for the even and odd degrees require separate arguments. The details are spelled out, in the odd parity case at any rate, to convince the skeptical reader, as well as the writer.

We begin with a few observations akin to those of [8, §16].

1.9. Let \( r_1, \ldots, r_n \) be a quasi-regular sequence, \( I = (r_1, \ldots, r_n) \), and \( s \) an element in \( R \).

(a) If \( |s| \) is odd and \( (I : _Rs) = I + (s) \), then \( (I^v : _Rs) = I^v + (s) \) for all \( v \geq 1 \).

(b) If \( |s| \) is even and \( (I : _Rs) = I \), then \( (I^v : _Rs) = I^v \) for all \( v \geq 1 \).

(a) We induce on \( v \), the basis \( v = 1 \) being the given. Assume that the desired equality holds for some integer \( v \geq 1 \). If \( rs \in I^{v+1} \subset I^v \), then by the induction hypothesis \( r \in I^v + (s) \). Thus, there is a polynomial \( F \) in \( R[X] = R[x_1, \ldots, x_n] \) with \( \deg(F) = v \) such that \( r = F(r_1, \ldots, r_n) + as \) for some \( a \in R \). Now, \( sF \in R[X] \) is such that \( sF(r_1, \ldots, r_n) = s(r - as) = sr = \pm rs \in I^{v+1} \); since \( \deg(F) = v \) and the sequence \( r_1, \ldots, r_n \) is quasi-regular, according to 1.5 one has \( sF \in IR[X] \), that is to say, \( sc \in I \) for each coefficient \( c \) of the polynomial \( F \). Thus, \( c \in (I : _Rs) = I + (s) \), so that \( F \in (I + (s))R[X] \) and \( r = F(r_1, \ldots, r_n) + as \in (I + (s))I^v + (s) = I^{v+1} + (s) \), which is the desideratum.

The proof of (b) is analogous.
1.10. Proof of Proposition 1.7. We induce on $n$, the cardinality of the sequence $r_1, \ldots, r_n$; the basis step $n = 1$ is a straightforward verification. The induction hypothesis is that any regular sequence of length $n$ is quasi-regular.

Let $r_1, \ldots, r_{n+1}$ be a regular sequence; since the sequence $r_1, \ldots, r_n$ is regular as well, it must be quasi-regular. Set $I = (r_1, \ldots, r_n)$, $J = (r_1, \ldots, r_{n+1})$, and set $R[X] = R[x_1, \ldots, x_n]$. By 1.5, it suffices to establish that if a polynomial $F \in R[x_1, \ldots, x_{n+1}] = R[X, x_{n+1}]$, with $\deg(F) = v$, is such that $F(r_1, \ldots, r_{n+1}) = 0$, then $F \in JR[X, x_{n+1}]$. We may assume that $F \not\in R[X]$.

Suppose that $|x_{n+1}| = |r_{n+1}|$ is odd; the regularity of $r_{n+1}$ on $R/(r_1, \ldots, r_n)$ translates to the equality $(I: R r_{n+1}) = I + (r_{n+1})$. Now we induce on $v = \deg(F)$; the basis step $v = 0$ is tautological. Assume that the desired conclusion holds for all polynomials of external degree $v$ or less. If $\deg(F) = v + 1$, then $F = G + x_{n+1}H$, with $G, H \in R[X]$, $\deg(G) = v + 1$, and $\deg(H) = v$. Now

$$G(r_1, \ldots, r_n) + r_{n+1}H(r_1, \ldots, r_n) = F(r_1, \ldots, r_{n+1}) = 0,$$

so that by 1.9.a, one has $H(r_1, \ldots, r_n) \in (I^{v+1}; R r_{n+1}) = I^{v+1} + (r_{n+1})$. Therefore, there is a polynomial $\tilde{H} \in R[X]$ with $\deg(\tilde{H}) = v + 1$ such that $H(r_1, \ldots, r_n) = \tilde{H}(r_1, \ldots, r_n) + ar_{n+1}$ for some element $a \in R$. For $\tilde{G} = G + r_{n+1}\tilde{H} \in R[X]$ we find

$$\tilde{G}(r_1, \ldots, r_n) = G(r_1, \ldots, r_n) + r_{n+1}\tilde{H}(r_1, \ldots, r_n)$$
$$= G(r_1, \ldots, r_n) + r_{n+1}(H(r_1, \ldots, r_n) - ar_{n+1})$$
$$= G(r_1, \ldots, r_n) + r_{n+1}H(r_1, \ldots, r_n)$$
$$= F(r_1, \ldots, r_{n+1}) = 0.$$

Since $r_1, \ldots, r_n$ is quasi-regular, $\tilde{G} \in IR[X] \subset JR[X, x_{n+1}]$, so that $G = \tilde{G} - r_{n+1}\tilde{H} \in JR[X, x_{n+1}]$. Now we attend to $H$: The polynomial $r_{n+1}H \in R[X]$ is such that $r_{n+1}H(r_1, \ldots, r_n) \in r_{n+1}(I^{v+1} + (r_{n+1})) = r_{n+1}I^{v+1} \subset I^{v+1}$; the quasi-regularity of $r_1, \ldots, r_n$ yields $r_{n+1}H \in IR[X]$, that is to say, $r_{n+1}c \in I$ for each coefficient $c$ of $H$; hence $c \in (I: R r_{n+1}) = I + (r_{n+1}) = J$, and $H \in JR[X]$. Therefore, $F = G + x_{n+1}H \in JR[X, x_{n+1}]$, which is what we seek.

The case where $|x_{n+1}|$ is even is handled analogously. \qed

Now we prove Proposition 1.8.

1.11. Proof of Proposition 1.8. Set $I = (r_1, \ldots, r_n)$. The hypothesis ensures that

$$\left( \bigcap_{\nu \geq 1} I^\nu \right) \subseteq \bigcap_{\nu \geq 1} \text{Jac}(R_0)R_\nu = 0 \quad \text{for all } i \geq 0,$$

where the equality on the right is due to Krull’s intersection theorem, cf. [8, (8.10)]; thus $\bigcap_{\nu \geq 1} I^\nu = 0$. We claim that $r_1$ is regular.

Indeed, suppose that $|r_1|$ is odd, and that $sr_1 = 0$ with $s \in R$. It suffices to prove the following statement: For each integer $v \geq 1$, there is an element $a_v \in R$ and a polynomial $F_v \in R[x_2, \ldots, x_n]$ with

$$\deg(F_v) = v \quad \text{and} \quad s = a_v r_1 + F_v(r_2, \ldots, r_n).$$

For then, modulo $r_1$, one has $s \in \bigcap_{\nu \geq 1} I^\nu = 0$; thus $s \in (r_1)$ as desired.

We resort to an induction on $v$ to find the $a_v$’s and $F_v$’s. Since $sr_1 = 0$, 1.5 applied to the polynomial $sx_1 \in R[X] = R[x_1, \ldots, x_n]$ yields $s \in I$; say $s = a_1 r_1 + \sum_{i \geq 2} a_i r_i$. Then, $a_1$ and $F_1 = \sum_{i \geq 2} a_i x_i$ form the basis of the induction.
Assume that $a_v$ and $F_v$ with the desired properties have been found, for some integer $v \geq 1$. Then

$$0 = r_1 s = r_1 (a_v r_1 + F_v(r_2, \ldots, r_n)) = r_1 F_v(r_2, \ldots, r_n).$$

By 1.5, the coefficients of the polynomial $x_1 F_v \in R[X]$ are in $I$. Observe that since $F_v \in R[x_2, \ldots, x_n]$ the set of coefficients of the polynomials $F_v$ and $x_1 F_v$ coincide, which allows us to deduce that the coefficients of $F_v$ are all in $I$. Thus, after a spot of rearrangement, we can find a polynomial $F_{v+1} \in R[x_2, \ldots, x_n]$ and an element $a \in R$ such that $F_v(r_2, \ldots, r_n) = ar_1 + F_{v+1}(r_2, \ldots, r_n)$, so that

$$s = a_v r_1 + F_v(r_2, \ldots, r_n) = (a_v + a)r_1 + F_{v+1}(r_2, \ldots, r_n).$$

Setting $a_{v+1} = a$ completes the induction step.

The proof in the case when $|r_1|$ is even is similar, and a tad simpler.

Now, set $S = R/(r_1)$. To complete the proof the proposition, it suffices to prove that the (images of the) elements $r_2, \ldots, r_n$ are a quasi-regular sequence in $S$. Once again we are faced with two cases; we content ourselves by considering the situation when $|r_1|$ is odd.

Suppose that $F(r_2, \ldots, r_n) = 0$ for an $F \in S[x_2, \ldots, x_n]$ with $\deg(F) = v$. Pick an element $\tilde{F} \in R[x_2, \ldots, x_n]$, with $\deg(\tilde{F}) = v$, such that it maps to $F$ under the canonical surjection $R[x_2, \ldots, x_n] \rightarrow S[x_2, \ldots, x_n]$. Therefore, $\tilde{F}(r_2, \ldots, r_n) = 0$ translates to $\tilde{F}(r_2, \ldots, r_n) = ar_1$ for some $a \in R$; in particular $ar_1 \in I^v$. Since $r_1, \ldots, r_n$ is quasi-regular, a repeated application of 1.5.ii yields $a \in (r_1) + I^{v-1}$, so we may find a polynomial $G \in R[X] = R[x_1, \ldots, x_n]$ with $\deg(G) = v - 1$ so that $a = a'r_1 + G(r_1, \ldots, r_n)$ for some $a' \in R$. Hence, $\tilde{F}(r_2, \ldots, r_n) = ar_1 - G(r_1, \ldots, r_n)r_1$, that is to say the polynomial $\tilde{F} - Gx_1$ evaluated at $r_1, \ldots, r_n$ is zero. Thus, by 1.5, we obtain $\tilde{F} - Gx_1 \in IR[X]$. Remains to note that $\tilde{F}$ does not involve any non-zero terms with the variable $x_1$ so that $\tilde{F} \in IR[X]$ and hence its image $F \in JS[X]$, where $J$ is the ideal in $S$ generated by $r_2, \ldots, r_n$. \hfill \Box

\section{Tate extensions}

In the theory of commutative rings, regularity of a sequence is related to the acyclicity of the associated Koszul complex. The analogous fact, in the context of graded-commutative algebras, was established by Tate [10]. This result is recalled below, as formulated by Avramov [1, (6.1.7)]; we refer the reader to loc. cit. and [5], for unexplained notation, definitions, and basic constructions concerning DG algebras.

\subsection{2.1. Let $A$ be a DG algebra, let $w = w_1, \ldots, w_n$ be classes in $H(A)$, let $z_1, \ldots, z_n$ be cycles such that $\text{cls}(z_i) = w_i$ for each $i$, and let $A(U) = A(u_1, \ldots, u_n | \partial(u_i) = z_i)$. Implications $(i) \implies (ii) \implies (iii)$ hold among the following conditions:

(i) $w_1, \ldots, w_n$ is a regular sequence in $H(A)$;

(ii) the canonical map $H(A) \rightarrow H(A(U))$ is an isomorphism;

(iii) the canonical map $H(A) \rightarrow H(A(U))$ is surjective.

The conditions are equivalent if $|w_i| \geq 1$ for all $i$, or each $H_j(A)$ is noetherian over $H_0(A)$ and if $|w_i| = 0$, then $w_i \in \text{Jac}(H_0(A))$.

Combining the result above with Proposition 1.8, we arrive at
2.2. Theorem. Let $A$ be a DG algebra, let $w = w_1, \ldots, w_n$ be classes in $H(A)$, such that $|w_i| \geq 1$ for all $i$, or each $H_i(A)$ is noetherian over $H_0(A)$ and if $|w_i| = 0$, then $w_i \in \text{Jac}(H_0(A))$. Let $z_1, \ldots, z_n$ be cycles such that $\text{cls}(z_i) = w_i$ for each $i$, and let $A(U) = A(u_1, \ldots, u_n \mid \partial(u_i) = z_i)$.

The following conditions are equivalent.

(i) The morphism of graded-commutative algebras is an isomorphism:

$$\frac{H(A)}{(w)} H(A)[x_1, \ldots, x_n] \rightarrow \text{gr}_{(w)}(H(A)) \quad \text{with} \quad x_i \mapsto w_i;$$

(ii) The canonical map $\frac{H(A)}{(w)} H(A) \rightarrow H(A(U))$ is an isomorphism;

(iii) the canonical map $H(A) \rightarrow H(A(U))$ is surjective. $\square$

With additional hypotheses, one arrives at a more gratifying result; it generalizes a theorem of Blanco, Majadas, and Rodicio, [4, Theorem 1] who prove the special case where $A$ is the Koszul complex on a set of elements over a commutative ring and each class $w_i$ has degree 1.

2.3. Theorem. Let $A$ be a DG algebra, let $w_1, \ldots, w_n$ be classes in $H(A)$, such that $|w_i| \geq 1$ for all $i$. Let $z_1, \ldots, z_n$ be cycles such that $\text{cls}(z_i) = w_i$ for each $i$, and let $A(U) = A(u_1, \ldots, u_n \mid \partial(u_i) = z_i)$.

The following are equivalent:

(i) The morphism of $H_0(A)$-algebras $H_0(A)[y_1, \ldots, y_n] \rightarrow H(A)$ with $y_i \mapsto w_i$, is an isomorphism;

(ii) the DG algebra $A(U)$ is acyclic.

Proof. (i) $\implies$ (ii) The sequence $w_1, \ldots, w_n$ is regular on $H(A)$; now invoke 2.1.

(ii) $\implies$ (i) Since $|z_i| \geq 1$, the map $H(A) \rightarrow H(A(U))$ is an isomorphism in degree 0, hence surjective, and so, thanks to 2.1, we deduce that the sequence $w_1, \ldots, w_n$ is regular on $H(A)$, and that $(w_1, \ldots, w_n)$ is the ideal of elements of positive degree in $H(A)$. In particular, the map in question $H_0(A)[y_1, \ldots, y_n] \rightarrow H(A)$ is surjective; injectivity is asserted by Proposition 1.6, for, according to Proposition 1.7, a regular sequence is quasi-regular. $\square$

The following corollary was suggested to me by J.P.C. Greenlees; it may be viewed as the graded-commutative analogue of the Auslander-Buchsbaum test for detecting regular local rings: A noetherian local ring $(R, m, k)$ is regular if and only if, for elements $r_1, \ldots, r_n \in m$ that map to a basis for the $k$-vector space $m/m^2$, the Koszul complex $R(u_1, \ldots, u_n \mid \partial(u_i) = r_i)$ is acyclic.

2.4. Corollary. Let $R$ be a graded-commutative algebra with $R_0$ a field $k$, let $m$ be the kernel of the canonical surjection $R \rightarrow k$, and assume that the $k$-vector space $m/m^2$ is finitely dimensional. Let $r_1, \ldots, r_n \in R$ map to a basis for $m/m^2$, and set $R(U) = R(u_1, \ldots, u_n \mid \partial(u_i) = r_i)$. The following conditions are equivalent.

(i) $R$ is isomorphic to a free graded-commutative algebra over $k$;

(ii) The morphism of graded-commutative algebras $k[y_1, \ldots, y_n] \rightarrow R$, where $y_j \mapsto r_j$, is an isomorphism;

(iii) The DG algebra $R(U)$ is acyclic.

Proof. (ii) $\implies$ (i) is tautological; (ii) $\iff$ (iii) is a special case of Theorem 2.3.
(i) \(\implies\) (ii) Let \(R \xrightarrow{\cong} k[X]\), and let \(\alpha\) be the composed \(k[Y] = k[y_1, \ldots, y_n] \to R \xrightarrow{\cong} k[X]\). The choice of the \(r_i\) ensures that the induced map of graded \(k\)-vector spaces \(kY \to kX\) is an isomorphism, therefore, by the (graded version of) Nakayama’s lemma, \(\alpha\) is surjective; let \(J\) be its kernel. The universal property of free algebras provides a morphism of \(k\)-algebras \(\sigma: k[X] \to k[Y]\) such that \(\alpha \circ \sigma = \text{id}\). In particular, \(\text{Tor}^\alpha (k, k) \circ \text{Tor}^\sigma (k, k)\) is the identity map of \(\text{Tor}^{k[X]} (k, k)\), and hence \(\text{Tor}^\alpha (k, k)\) is surjective. Thus, \(2, (1.6)\) applied to \(\alpha\) and \(\beta: k[X] \to k\), yields an exact graded \(k\)-vector spaces \(0 \to J/(Y)J \to kY \to kX \to 0\). Since the map \(kY \to kX\) is an isomorphism, \(J = (Y)J\), so that another application of Nakayama’s lemma yields \(J = 0\), that is to say, \(\alpha\) is an isomorphism. \(\square\)

3. LARGE HOMOMORPHISMS

A surjective local homomorphism \(\phi: (R, m, k) \to (S, n, k)\) is large if the induced map \(\text{Tor}^\phi (k, k): \text{Tor}^R (k, k) \to \text{Tor}^S (k, k)\) is surjective. For example, the surjection \(R \to k\) is large, as is a split local homomorphism \(\phi: R \to S\), that is to one, one which admits a section \(\rho: S \to R\) such that \(\phi \circ \rho = \text{id}^S\).

This paper originates in an attempt to deal with the following question; confer \([1, \S 6.3]\) for information concerning acyclic closures.

**Problem.** Let \(R \to S\) be a large homomorphism and let \(R(U)\) be the acyclic closure of \(S\) over \(R\). If \(U_i = \emptyset\) for all \(i \gg 0\), then is it true that \(U_i = 0\) for \(i \geq 3\) ?

This problem may be viewed as a DG algebra version of a conjecture of Quillen on the vanishing of cotangent homology \([9, (5.6)]\), specialized to large homomorphisms. When the characteristic of the field \(k\) is zero, the question above coincides with Quillen’s conjecture for the particular case of large homomorphisms; cf. \([9, (9.5)]\) or \([2, (3.3.2)]\).

As mentioned in the introduction, the question above has an affirmative answer for the large homomorphism \(R \to k\); confer \([1, \S 7.3]\). In fact, in this case, it suffices to assume that \(U_i = 0\) for some integer \(i \geq 1\). Avramov and the author \([3]\), have settled the question, once again in the affirmative, in the case when the homomorphism \(R \to S\) is split. Note that there are large homomorphisms for which \(\sup\{i \mid U_i \neq \emptyset\}\) is either 1 or 2, cf. \([10, \text{Theorem 4}]\), or \([2, (4.2)]\).

Not all large homomorphisms are of the preceding types, cf. \([7]\), so it seems worth while to report the following:

**3.1. Theorem.** Let \(R \to S\) be a large homomorphism and let \(R(U)\) be the acyclic closure of \(S\) over \(R\). If \(\sup\{i \mid U_i \neq \emptyset\} = s < \infty\), then either \(s = 1\) or \(s\) is even.

The proof requires some preparation, and is deferred to the end of this section.

In the sequel, for a DG algebra \(A\) we write \(\sup(A)\) for \(\sup\{i \mid H_i (A) \neq 0\}\); in particular, \(\sup(A) = -\infty\) if \(H(A) = 0\).

Let \(A \to A(x) \mid \partial (x) = z\) be the Tate extension with \(|x| = d\) and \(w = \text{cls}(z)\).

3.2. When \(d\) is odd, there is an exact sequence of chain maps

\[
0 \to A \to A(x) \to A \xrightarrow{\nu} 0 \quad \text{where} \quad \nu (a + xb) = b
\]

and the corresponding homology long exact sequence is

\[
\cdots \to H_{i-d}(A) \xrightarrow{\partial} H_i(A) \to H_i(A(x)) \to H_{i-1}(A) \to \cdots
\]

In particular, if \(\sup(A) < \infty\), then \(\sup(A(x)) < \infty\); the converse holds if \(z \in mZ(A)\).
Iterating this, we arrive at: Let $z_1, \ldots, z_n$ be cycles in a DG algebra $A$ and let $A(X) = A(x_1, \ldots, x_n \mid \partial(x_i) = z_i)$. If $|x_i|$ is odd for each $i$ and $\sup(A) < \infty$, then $\sup(A(X)) < \infty$. The converse holds if $z_i \in mZ(A)$ for each $i$.

3.3. When $d$ is even, there is an exact sequence of chain maps

$$0 \to A \to A(x) \xrightarrow{\varepsilon} A(x) \to 0$$

where $v \left( \sum_j a_j x^{(j)} \right) = \sum_j a_j x^{(j-1)}$

and this results in the homology long exact sequence

$$\cdots \to H_{i-d}(A(x)) \to H_i(A) \to H_{i-d}(A(x)) \to H_{i-d-1}(A(x)) \to \cdots$$

Therefore, if $\sup(A(x)) < \infty$, then $\sup(A) < \infty$.

An iteration leads to: Let $z_1, \ldots, z_n$ be cycles in a DG algebra $A$ and let $A(X) = A(x_1, \ldots, x_n \mid \partial(x_i) = z_i)$. If $|x_i|$ is even for each $i$ and $\sup(A(X)) < \infty$, then $\sup(A) < \infty$.

A semi-free $\Gamma$-extension $A(X)$ of a DG algebra $X$ is a DG algebra obtained by a successive adjunction of exterior and divided powers variables, cf. [1, (6.1)]. We shall require the following facts concerning acyclic closures of large homomorphisms.

3.4. Let $(R, m, k) \to (S, n, k)$ be a large homomorphism, and let $R\langle U \rangle$ be its acyclic closure. Then

(a) $R(U)$ is minimal, that is to say, $\partial(R(U)) \subseteq mR(U)$;

(b) there is a semi-free $\Gamma$-extension $R(U) \to R(U, \tilde{U})$, with $R(U, \tilde{U})$ the acyclic closure of $k$ over $R$.

The first statement is a result of Avramov and Rahbar-Rochandel [7, (2.5)], also cf. [2, (2.7)]. Given this fact, the second assertion follows from [7, (1.1.2)].

For the rest of this paper, it is convenient to adopt the following terminology: A partial acyclic closure of $S$ over $R$ is a DG $\Gamma$-algebra $R\langle U \rangle$ which admits a semi-free $\Gamma$-extension $R\langle U \rangle \langle W \rangle$ which is an acyclic closure of $S$ over $R$.

3.5. Proposition. Let $R \to S$ be a large homomorphism, and let $R\langle V \rangle \langle X \rangle$ be a partial acyclic closure of $S$ over $R$ with $\text{card}(X) < \infty$. If $\sup(R(V, X)) < \infty$, then $\sup(R(V)) < \infty$.

Proof. It is enough to prove the result when $X = \{x\}$. The case when the degree of $x$ is even is settled by 3.3; when $|x| = 1$, since $R\langle V, x \rangle$ is a partial acyclic closure of $S$ over $R$ it must be that $\partial(x) \in m$, so 3.2 yields the desired result. Thus, it remains to consider the case when $|x| \geq 3$ and odd.

Since $R \to S$ is large, the acyclic closure of $S$ over $R$ extends to an acyclic closure of $k$ over $R$, 3.4, so that there is a semi-free $\Gamma$-extension $R(V, x) \to R(Y)$ with $R(Y)$ the acyclic closure of $k$ over $R$. Adjoining the set of variables $W = Y_1 \setminus V_1$, we obtain a diagram of semi-free $\Gamma$-extensions

$$R(V) \langle W \rangle \to R(V, x) \langle W \rangle \to R(Y)$$

Set $A = R(V, W)$, and observe that the DG $\Gamma$-algebra $R(V, x, W)$ may be obtained by adjoining $x$ to $A$, that is to say, $R(V, x, W) = A(x)$. Thanks to 3.2, it suffice to prove that $\sup(A) < \infty$; by the same token, since $\sup(R(V, x)) < \infty$ we obtain that $\sup(A(x)) < \infty$. By construction, one has $H_0(A) = k$, and

$$Z_{>1}(A) \subseteq Z_{>1}(R(Y)) \cap A \subseteq \partial(R(Y)) \cap A \subseteq mR(Y) \cap A = mA$$
where, reading from the left, the second inclusion holds since \( R(Y) \) is acyclic, the third since it is minimal 3.4, and the last since \( R(Y) \) is a free \( A \)-module. Now we have all the prerequisites to invoke [6, Lemma 2], which yields \( \sup(A) < \infty \). □

**Proof of Theorem 3.1.** We may assume that \( s \geq 3 \). Set \( V = \bigcup_{i=1}^{s-1} U_i \). Since \( R(U) = R(V) \langle U_s \rangle \) is acyclic, with homology \( S \), Theorem 2.3 yields \( H(R(V)) \) is isomorphic to \( S[w_1, \ldots, w_n] \), where \( |w_i| = s - 1 \).

If \( s \) is odd, then \( |w_i| \) is even for each \( i \), so that \( \sup(R(V)) = \infty \). On the other hand, thanks to Proposition 3.5, the acyclicity of \( R(V) \langle U_s \rangle \) entails \( \sup(R(V)) < \infty \); a contradiction. □

**References**


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