MAXIMAL MINIMAL RESOLUTIONS

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Dedicated to Luchezar Avramov on the occasion of his fiftieth birthday

1: Introduction

A free resolution of a module \( M \) over a ring \( R \) is a complex of free \( R \)-modules

\[
\cdots \to F_2 \to F_1 \to F_0 \to 0
\]

that is exact everywhere except at \( F_0 \), where the cokernel is \( M \). Such a resolution always exists. If \( R \) is a noetherian \( \mathbb{N} \)-graded ring, with \( R_0 \) a field, and \( M \) is a finitely generated \( \mathbb{Z} \)-graded \( R \)-module, then a resolution may be constructed in a minimal way in which each \( F_i \) is graded and each differential is homogeneous of degree 0. The rank of the \( i \)th free module in a minimal free resolution of \( M \) is an invariant of \( M \), called the \( i \)th Betti number of \( M \) and denoted \( \beta_i^R(M) \). Likewise, \( \beta_{ij}^R(M) \), the number of elements of degree \( j \) in a minimal set of homogeneous generators of \( F_i \), is also an invariant of \( M \), called the \((i, j)\)th graded Betti number of \( M \).

In this paper we study modules with maximal graded Betti numbers. More precisely, consider a set \( \Pi \) consisting of pairs \((R, M)\) where \( R \) is a noetherian \( \mathbb{N} \)-graded ring, with \( R_0 \) a field, and \( M \) is a finitely generated graded \( R \)-module. Is there some \((R', M')\) \( \in \Pi \) such that \( \beta_{ij}^{R'}(M') \geq \beta_{ij}^R(M) \) for every \( i \) and \( j \) and every \((R, M)\) \( \in \Pi \)? There are trivial examples in which the answer is yes (e.g., \( \Pi \) has only one element) or no (e.g., \( \Pi \) consists of all such pairs). Theorems 1 and 2 below give affirmative answers for certain sets \( \Pi \) defined by conditions on the Hilbert series and the depths of \( R \) and \( M \). Theorem 3 allows one to compute the maximal graded Betti numbers for the pairs in the sets considered in Theorems 1 and 2.

Throughout this paper, \( k \) denotes a field and \( Q \) the polynomial ring \( k[x_1, \ldots, x_n] \) with the usual \( \mathbb{N} \)-grading given by \( \deg x_i = 1 \). The Hilbert series of a finitely generated graded \( Q \)-module \( M \) is \( H_M(s) = \sum \dim_k M_is^i \). We define \( d \)-lexicographic ideals and submodules in Section 2. The Poincaré series of an \( R \)-module \( M \) is \( P_M(s; t) = \sum \beta_{ij}^R(M)s^jt^i \). We write \( \preceq \) for coefficient-wise inequality of Laurent series with coefficients in \( \mathbb{Z} \).
Theorem 1. Fix power series \( h_1(s), h_2(s) \in \mathbb{Z}[[s]] \), and a non-negative integer \( d \).

Let \( \Pi \) be the set of pairs \((Q/a, Q/b)\) where \( a \) is a homogeneous ideal with \( H_{Q/a}(s) = h_1(s) \) and \( \text{depth} Q/a \geq d \), and \( b \supseteq a \) is a homogeneous ideal with \( H_{Q/b}(s) = h_2(s) \) and \( \text{depth} Q/b \geq d \).

If \( \Pi \) is non-empty, then there is a pair \((R', S') \in \Pi \) such that \( P_{S'}^R(s; t) \preceq P_{S'}^{R'}(s; t) \) for each \((R, S) \in \Pi \). We may take \( R' = Q/a' \) and \( S' = Q/b' \) where \( a' \subseteq b' \subseteq Q \) are \( d \)-lexicographic ideals.

When \( h_1(s) = H_Q(s) \) (so that \( Q/a = Q \)), \( d = 0 \), and the characteristic of \( k \) is 0, the theorem has been proven independently by Hulett [17] and by Bigatti [8]. When \( h_2(s) = 1 \) (so that \( Q/b = k \)) and \( d = 0 \), the theorem has been proven by Peeva [24]. We were motivated to consider the present form of Theorem 1 by Peeva’s theorem; however, our proof is based on the techniques in [22].

Theorem 2. Fix a power series \( h(s) \in \mathbb{Z}[[s]] \), a non-negative integer \( d \), and a finitely generated graded free \( Q \)-module \( F \).

Let \( \Pi \) be the set of pairs \((Q, F/M)\), where \( M \subseteq F \) is a homogeneous submodule such that \( H_{F/M}(s) = h(s) \) and \( \text{depth} F/M \geq d \).

If \( \Pi \) is non-empty, then there exists a pair \((Q, F/M') \in \Pi \) such that \( P_{F/M'}^Q(s; t) \preceq P_{F/M}^Q(s; t) \) for each \((Q, F/M) \in \Pi \). We may take \( M' \) to be a \( d \)-lexicographic submodule of \( F \).

Note that in Theorem 1, the ring \( R = Q/a \) varies while the ring \( Q \) is fixed in Theorem 2.

When \( d = 0 \) and the characteristic of \( k \) is 0, the theorem has been proven by Hulett [18]. Pardue extended Hulett’s theorem to positive characteristic and also proved a version of the theorem for local rings [22]. Note that the dimension of \( F/M \) depends only on \( h(s) \).

The first parts of Theorems 1 and 2 are existential in nature: they assure us that there are maximal Poincaré series in the corresponding sets, but say nothing about what these are. In the case of Theorem 2, the maximal Poincaré series may be obtained from the Eliahou-Kervaire resolution of stable ideals [12]:

Theorem (Eliahou, Kervaire). Let \( a \) be a stable ideal in \( Q \).

The Poincaré series of \( R = Q/a \) as a \( Q \)-module is given by

\[
P_R^Q(s; t) = 1 + \sum_{a \in G(a)} s^{\deg a} t(1 + st)^{\max(a)} - 1,
\]

where \( G(a) \) is the minimal set of monomial generators of \( a \).

Stable ideals and “max” of a monomial are defined in Section 4. A \( d \)-lexicographic ideal is stable. If \( M' \) is a \( d \)-lexicographic submodule in Theorem 2, then \( F/M' \) is isomorphic to a direct sum of cyclic \( Q \)-modules, each of which has a \( d \)-lexicographic ideal as its annihilator.
Thus, Eliahou and Kervaire’s formula allows one to compute the maximal Poincaré series that arises in Theorem 2. Theorem 3 allows one to compute the maximal Poincaré series that arises in Theorem 1.

**Theorem 3.** Let \( a \) and \( b \) be stable ideals such that \( a \subseteq b \cap (x_1, \ldots, x_n)^2 \) and consider \( S = Q/b \) as a module over \( R = Q/a \) via the canonical surjection.

The Poincaré series of \( S \) as an \( R \)-module is given by

\[
P^R_S(s; t) = \frac{1 + t \left( \sum_{b \in G(b)} s^{\deg b} (1 + st)^{\max(b)-1} - \sum_{c \in G(a) \cap G(b)} s^{\deg c} (1 + st)^{\max(c)} \right)}{1 - t^2 \sum_{a \in G(a)} s^{\deg a} (1 + st)^{\max(a)-1}},
\]

where \( G(a) \) and \( G(b) \) are minimal sets of monomial generators of \( a \) and \( b \) respectively.

The hypothesis that \( a \subseteq (x_1, \ldots, x_n)^2 \) is not stringent. If the degree-one part of \( a \) is non-trivial then it is spanned by variables \( x_1, \ldots, x_k \) and \( b \) contains these variables as well. In this case we replace \( Q \) by \( Q/(x_1, \ldots, x_k) \) before applying Theorem 3.

In the special case where \( b = (x_1, \ldots, x_n) \) (so that \( S = k \)), the Poincaré series was obtained by Aramova and Herzog [1] and by Peeva [24], using different techniques. The proof of Theorem 3 builds on the works of Eliahou and Kervaire, of Peeva [24], who endowed their resolution with an algebra structure, and of Avramov [5], who showed how to use multiplicative structures on resolutions to compute Betti numbers.

Section 2 collects notation and definitions needed for the rest of the paper and also states the conditions for the existence of \( d \)-lexicographic submodules. In Section 3, we prove Theorems 1 and 2 using deformation techniques. Section 4 reviews some results of Eliahou and Kervaire and of Peeva on stable ideals. These are necessary inputs for the techniques used in Section 5 to prove Theorem 3. In Section 6, we show how to modify our results for local rings. Section 7 discusses related problems for Gorenstein rings.

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### 2: Notation and Definitions

We use the following notations throughout this paper. \( k \) is a field and \( Q = k[x_1, \ldots, x_n] \) is a polynomial ring over \( k \). We consider \( Q \) as an \( \mathbb{N} \)-graded ring with \( \deg x_i = 1 \). The ideal \( m = (x_1, \ldots, x_n) \) is the graded maximal ideal of \( Q \). \( F \) is a free \( \mathbb{Z} \)-graded \( Q \)-module with a fixed basis \( f_1, \ldots, f_r \) such that \( f_i \) is homogeneous of degree \( d_i \) and \( d_1 \leq \cdots \leq d_r \). Any reference to a submodule of \( F \) applies to an ideal as well, since \( F \) may be \( Q \).

A **monomial** of \( F \) is an element of the form \( x^\mu f_i = x_1^{\mu_1} \cdots x_n^{\mu_n} f_i \). A **monomial submodule** \( M \subseteq F \) is a submodule generated by monomials. We consider two monomial orders on the monomials of \( F \): lexicographic and reverse lexicographic order.

The **lexicographic order** on monomials of \( F \) is given by \( x^\mu f_i \succ_{\text{lex}} x^\nu f_j \) if either

1. \( i < j \), or
2. \( i = j \) and the first non-zero entry in \( \mu - \nu \) is positive.
The dimension $\ell$ lexicographic subspace of elements of degree $i$ is the monomial subspace of $F_i$ spanned by the first $\ell$ monomials in the lexicographic order. A submodule $L$ of $F$ is a lexicographic submodule if it is graded and if $L_i$ is a lexicographic subspace of $F_i$ for every $i$. For a non-negative integer $d$, a monomial submodule $L$ of $F$ is a $d$-lexicographic submodule if $x_n, \ldots, x_{n-d+1}$ is a regular sequence on $F/L$ and the image of $L$ in $F/(x_n, \ldots, x_{n-d+1})F$ is a lexicographic submodule of the $Q/(x_n, \ldots, x_{n-d+1})$-module $F/(x_n, \ldots, x_{n-d+1})F$. Note that a 0-lexicographic submodule is simply a lexicographic submodule.

An important property of lexicographic order, which is used extensively in this paper, is the existence of $d$-lexicographic submodules of $F$ under the hypotheses given in the proposition below.

Recall that the Hilbert series of a finitely generated $\mathbb{Z}$-graded $Q$-module $M = \bigoplus_i M_i$ is the Laurent series $H_M(s) = \sum \dim_k M_i s^i$.

**Proposition 4.** Let $h(s) \in \mathbb{Z}((s))$ be a Laurent series and $d$ a non-negative integer. There is a submodule $M \subseteq F$ such that $F/M$ has Hilbert series $h(s)$ and $F/M$ has depth at least $d$ if and only if there is a $d$-lexicographic submodule $L$ of $F$ such that $F/L$ has Hilbert series $h(s)$. The $d$-lexicographic submodule of $F$ with Hilbert series $h(s)$ is unique.

When $d = 0$ and $F = Q$, the proposition is a well-known theorem of Macaulay [19].

**Proof.** Since the depth of $F/L$ is at least $d$ it suffices to prove that if there is such an $M$, there must be such a $d$-lexicographic submodule of $F$. When $d = 0$ this is proven by Hulett [18].

Assume now that $d > 0$ and that $k$ is an infinite field. Then there is a sequence of linear forms $\ell_n, \ldots, \ell_{n-d+1}$ that is regular on $F/M$. After making a linear change of variables in $Q$, we may assume that $x_n, \ldots, x_{n-d+1}$ is a regular sequence on $F/M$. Let $\overline{Q} = Q/(x_n, \ldots, x_{n-d+1}) = k[x_1, \ldots, x_d]$, and $\overline{F}$ be the free $\overline{Q}$-module $F/(x_n, \ldots, x_{n-d+1})F$, and $\overline{M}$ be the image of $M$ in $\overline{F}$. Since $x_n, \ldots, x_{n-d+1}$ is a regular sequence on $F/M$, the Hilbert series of $\overline{F}/\overline{M}$ is $(1 - s)^d h(s)$. By Hulett’s theorem there is a lexicographic submodule $\overline{L} \subseteq \overline{F}$ such that $\overline{F}/\overline{L}$ has Hilbert series $(1 - s)^d h(s)$. Now, consider $Q$ as the subring $k[x_1, \ldots, x_d]$ of $Q$ and $F$ as a $Q$-submodule of $F$. Let $L$ be the $Q$-submodule of $F$ generated by $\overline{L}$. Then $F/L$ has Hilbert series $(1 - s)^{-d}(1 - s)^d h(s) = h(s)$. Since the image of $L$ in $\overline{F}$ is $\overline{L}$, $L$ is the $d$-lexicographic submodule that we seek.

Now assume that $k$ is finite. Let $k'$ be an infinite field containing $k$. The result in the last paragraph allows us to construct a $d$-lexicographic submodule $L'$ of the free $Q \otimes_k k'$-module $F \otimes_k k'$ such that $(F \otimes_k k')/L'$ has Hilbert series $h(s)$. The generators of $L'$ are monomials, so they are defined over $k$. Thus, we may take $L$ to be the submodule of $F$ generated by these monomials. Since $F/L$ has the same Hilbert series and depth as $(F \otimes_k k')/L'$, the $d$-lexicographic submodule that we seek is $L$.

The uniqueness of the $d$-lexicographic submodule is clear. \qed

The reverse lexicographic order on monomials of $F$ is given by $x^\mu f_i >_{revlex} x^\nu f_j$ if either

1. $\deg x^\mu f_i > \deg x^\nu f_j$, or
2. $\deg x^\mu f_i = \deg x^\nu f_j$, $\mu \neq \nu$, and the last nonzero entry of $\mu - \nu$ is negative, or
3. $\mu = \nu$, $f_i$ and $f_j$ have the same degree, and $i < j$. 
Let $>$ be either $>_\text{lex}$ or $>_\text{revlex}$. If $f \in F$ is the sum of $\alpha x^\mu f_i$, with $\alpha \in k^*$, and a $k$-linear combination of monomials that are smaller than $\alpha x^\mu f_i$ with respect to $>$, then $\alpha x^\mu f_i$ is called the initial term of $f$ with respect to $>$ and is denoted $\text{in}(f)$. The initial term of 0 is 0. If $N$ is a submodule of $F$, then the submodule of initial terms of $N$ is the submodule of $F$ generated by the initial terms of elements of $N$; it is denoted $\text{in}(N)$. We will also write $\text{in}_{\text{lex}}(N)$ or $\text{in}_{\text{revlex}}(N)$ to emphasize which order we are using.

Consider $GL(n)$ as the group of $k$-linear graded algebra automorphisms of $Q$, and let $GL(F)$ be the group of $Q$-linear graded automorphisms of $F$. Then $G = GL(n) \times GL(F)$ is an algebraic group that acts on $F$ through $k$-linear graded automorphisms that take submodules to submodules. Let $B$ be the subgroup of $G$ consisting of all automorphisms taking $f_i$ to a $Q$-linear combination of $f_1, \ldots, f_i$ and $x_i$ to a $k$-linear combination of $x_1, \ldots, x_i$. $B$ is a Borel group of $G$ and is naturally realized as upper triangular matrices. A submodule $N$ of $F$ is Borel-fixed if $\gamma(N) = N$ for every $\gamma \in B$. See [21] or [22] for the combinatorial description of a Borel-fixed submodule.

We will use the following two properties of the action of $G$ on $F$ and the reverse lexicographic order on monomials of $F$.

**Proposition 5.** Let $N$ be a submodule of $F$ and let $G$ and $B$ be as above. Let $>$ be either lexicographic or reverse lexicographic order. Then there is a Zariski-open set $G \subseteq G$ such that

1. $\text{in}(\gamma_1(N)) = \text{in}(\gamma_2(N))$ for all $\gamma_1, \gamma_2 \in G$, and
2. $\text{in}(\gamma(N))$ is a Borel-fixed submodule of $F$ for all $\gamma \in G$.

**Proof.** See Example I.7 and Proposition VII.1 of [21], or modify the proofs of Theorems 15.18 and 15.20 in [10], which are for the case $F = Q$. □

A form of Proposition 5 was first proved by Galligo for the action of $GL(n)$ on $\mathbb{C}\{x_1, \ldots, x_n\}$ [14]. Proposition 5 was first proved by Bayer and Stillman in the case $F = Q$ [7].

If $\gamma \in G$ as in Proposition 5, then $\text{in}(\gamma(N))$ is called the generic initial submodule of $N$ with respect to the chosen order, and is denoted $\text{Gin}(N)$ or, to emphasize the order used, $\text{Gin}_{\text{lex}}(N)$ or $\text{Gin}_{\text{revlex}}(N)$.

**Proposition 6 (Bayer and Stillman).** Let $N$ be a submodule of $F$. Then

$$\text{depth} F / \text{Gin}_{\text{revlex}}(N) = \text{depth} F / N.$$ 

**Proof.** See Theorem 15.13 in [10]. □

3: Maximal Betti Numbers

In this section we prove Theorems 1 and 2, and two corollaries to Theorem 1. We use a modification of a technique developed by Pardue in [22], which is based on deformations used by Hartshorne [16] and Reeves [25] in their studies of Hilbert schemes. The modification used here also has applications to Hilbert schemes, which are given in [23].
Theorem 1. Fix power series \(h_1(s), h_2(s) \in \mathbb{Z}[[s]]\), and a non-negative integer \(d\).

Let \(\Pi\) be the set of pairs \((Q/\mathfrak{a}, Q/\mathfrak{b})\) where \(\mathfrak{a}\) is a homogeneous ideal with \(H_{Q/\mathfrak{a}}(s) = h_1(s)\) and \(\text{depth}Q/\mathfrak{a} \geq d\), and \(\mathfrak{b} \supseteq \mathfrak{a}\) is a homogeneous ideal with \(H_{Q/\mathfrak{b}}(s) = h_2(s)\) and \(\text{depth}Q/\mathfrak{b} \geq d\).

If \(\Pi\) is non-empty, then there is a pair \((R', S') \in \Pi\) such that \(P^R_S(s; t) \preceq P^R_{S'}(s; t)\) for each \((R, S) \in \Pi\). We may take \(R' = Q/\mathfrak{a}'\) and \(S' = Q/\mathfrak{b}'\) where \(\mathfrak{a}' \subseteq \mathfrak{b}' \subseteq Q\) are \(d\)-lexicographic ideals.

Proof. Let \(\mathfrak{a}'\) and \(\mathfrak{b}'\) be \(d\)-lexicographic ideals such that \((R', S') \in \Pi\) where \(R' = Q/\mathfrak{a}'\) and \(S' = Q/\mathfrak{b}'\). The existence of these ideals is guaranteed by Proposition 4.

Let \(\mathfrak{a} \subseteq \mathfrak{b} \subseteq Q\) be any homogeneous ideals such that \((R, S) \in \Pi\), where \(R = Q/\mathfrak{a}\) and \(S = Q/\mathfrak{b}\). We prove the theorem in three steps.

Step 1: Reduction to the case where \(\mathfrak{a}\) and \(\mathfrak{b}\) are Borel-fixed ideals.

Extending the ground field does not change depths, Hilbert series, or Poincaré series, so we assume that \(k\) is infinite.

Set \(\tilde{\mathfrak{a}} = \text{Gin}_{\text{revlex}}(\mathfrak{a})\) and \(\tilde{\mathfrak{b}} = \text{Gin}_{\text{revlex}}(\mathfrak{b})\). Proposition 6 shows that \(\text{depth}Q/\tilde{\mathfrak{a}} = \text{depth}Q/\mathfrak{a}\) and \(\text{depth}Q/\tilde{\mathfrak{b}} = \text{depth}Q/\mathfrak{b}\). Moreover, the Hilbert series of \(\tilde{R} = Q/\tilde{\mathfrak{a}}\) and \(\tilde{S} = Q/\tilde{\mathfrak{b}}\) are \(h_1(s)\) and \(h_2(s)\) respectively; hence \((\tilde{R}, \tilde{S}) \in \Pi\).

The Gröbner basis deformation (see Proposition 8 of [23] or modify Theorem 15.17 in [10]) gives us a graded \(k[t]\)-algebra \(T\) and a graded \(T\)-module \(U\) such that

a) \(T\) and \(U\) are \(k[t]\)-flat;

b) \(T \otimes_{k[t]} (k[t]/t) \cong \tilde{R}\) and \(U \otimes_{k[t]} (k[t]/t) \cong \tilde{S}\); and

c) \(T \otimes_{k[t]} (k[t]/(t-\alpha)) \cong R\) and \(U \otimes_{k[t]} (k[t]/(t-\alpha)) \cong S\) for every \(\alpha \in k^*\).

Since graded Betti numbers are upper-semicontinuous in flat families, we obtain an inequality of Poincaré series: \(P^R_{S'}(s; t) \preceq P^R_{S'}(s; t)\).

Proposition 5 shows that the ideals \(\tilde{\mathfrak{a}}\) and \(\tilde{\mathfrak{b}}\) are Borel-fixed. Thus, we replace \(\mathfrak{a}\) and \(\mathfrak{b}\) with \(\tilde{\mathfrak{a}}\) and \(\tilde{\mathfrak{b}}\), and henceforth assume that \(\mathfrak{a}\) and \(\mathfrak{b}\) are Borel-fixed ideals. Finally, note that \(\mathfrak{a}'\) and \(\mathfrak{b}'\) are Borel-fixed ideals as well.

Step 2: Reduction to the case where \(d = 0\).

The ideal \(I\) is Borel-fixed and \(\text{depth}Q/I \geq d\) for each \(I = \mathfrak{a}, \mathfrak{b}, \mathfrak{a}'\) or \(\mathfrak{b}'\). By Corollary 15.25 in [10], \(x = x_n, x_{n-1}, \ldots, x_{n-d+1}\) is a regular sequence on \(R = Q/\mathfrak{a}\) and \(S = Q/\mathfrak{b}\), and on \(R' = Q/\mathfrak{a}'\) and \(S' = Q/\mathfrak{b}'\). This yields equalities of Poincaré series

\[ P^R_S(s; t) = P^{R/1}(s; t) \quad \text{and} \quad P^{R'}_{S'}(s; t) = P^{R'/1}(s; t). \]

Thus, we may replace \(Q\) with \(Q/(x)\) and assume that \(d = 0\).

Step 3: Moving to lexicographic ideals.

Note that the ideals \(\mathfrak{a}'\) and \(\mathfrak{b}'\) are lexicographic, since \(d = 0\).

This step uses some properties of the operation \(\phi\) introduced by Pardue in [22]. For a monomial ideal \(I\), we set \(\phi(I) = \text{in}_{\text{lex}}(\sigma_L(I^{(p)}))\), where \(I^{(p)}\) is the polarization of the monomial ideal in a polynomial ring \(P = k[z_{ij}]\); \(L\) is a general collection of linear forms \(\ell_{ij} \in Q\); and \(\sigma_L\) is the homomorphism from \(P\) to \(Q\) taking \(z_{ij}\) to \(\ell_{ij}\). For an integer \(m \geq 2\), the operation \(\phi^m(I)\) is defined inductively via \(\phi^m(I) = \phi(\phi^{m-1}(I))\). [22] also shows that

1) \(\phi\) is well defined, in the sense that \(\phi(I)\) is independent of the choice of generic \(L\);
(2) $Q/I$ and $Q/\phi(I)$ have the same Hilbert series;
(3) if $I \subseteq J$, then $\phi(I) \subseteq \phi(J)$;
(4) if $L$ is a lexicographic ideal, then $\phi(L) = L$; and
(5) if $Q/I$ and $Q/L$ have the same Hilbert series and $L$ is lexicographic, then $\phi^m(I) = L$
for sufficiently large $m$.

In particular, $\phi^m(a) = a'$ and $\phi^m(b) = b'$ for sufficiently large $m$. To complete the proof
of Theorem 1, we must show that $\beta_{ij}^{Q/a}(Q/b) \leq \beta_{ij}^{Q/\phi(a)}(Q/\phi(b))$ for every $i$ and $j$.
Let $\pi : P \to Q$ be the homomorphism taking $z_{ij}$ to $x_i$. Then $\pi(a^{(p)}) = a$ and $\pi(b^{(p)}) = b$.
Fröberg showed that the kernel of $\pi$ is generated by a sequence of linear forms that is
regular on both $P/a^{(p)}$ and $P/b^{(p)}$ \cite{13}. As in \cite{22}, when $\mathbb{L}$ is generic the kernel of $\sigma_\mathbb{L}$
is also generated by a sequence of linear forms regular on both $P/a^{(p)}$ and $P/b^{(p)}$. This
justifies the equalities below:

$$
\beta_{ij}^{Q/a}(Q/b) = \beta_{ij}^{P/a^{(p)}}(P/b^{(p)}) = \beta_{ij}^{Q/\sigma_\mathbb{L}(a^{(p)})}(Q/\sigma_\mathbb{L}(b^{(p)})) \leq \beta_{ij}^{Q/\phi(a)}(Q/\phi(b)) ,
$$

and the inequality follows from the fact that graded Betti numbers do not decrease on
going to the initial ideal, as was shown in Step 1.

\[ \square \]

**Corollary 7.** With notation as in Theorem 1, further assume that $d$ is positive and that $k$
is an infinite field. Then we may take the ideals $a'$ and $b'$ in Theorem 1 to be intersections
of prime ideals generated by linear forms.

**Proof.** By Theorem 1, it suffices to consider $R = Q/a$ and $S = Q/b$ where $a$ and $b$ are
$d$-lexicographic ideals. Let $a' = \sigma_\mathbb{L}(a^{(p)})$ and $b' = \sigma_\mathbb{L}(b^{(p)})$ where $\sigma_{\mathbb{L}}$ and $I^{(p)}$
are as in Step 3 of the proof of Theorem 1. As in that step, we have for every $i$ and $j$ that

$$
\beta_{ij}^{Q/a}(Q/b) = \beta_{ij}^{Q/a'}(Q/b') ,
$$

so that $P_{Q/a}'(s;t) = P_{Q/b}'(s;t)$.

Now, we show that $a'$ and $b'$ are each intersections of prime ideals generated by linear
forms. The ideals $a^{(p)}$ and $b^{(p)}$ are radical monomial ideals in the polynomial ring $P = k[z_{ij}]$ and are thus the homogeneous ideals of unions of reduced coordinate planes in the
projective space $\text{Proj} P$. Since $\sigma_\mathbb{L} : P \to Q$ gives an inclusion of $\text{Proj} Q$ in $\text{Proj} P$ as a
general $n - 1$ dimensional linear space, the subschemes $\text{Proj} Q/a'$ and $\text{Proj} Q/b'$ of $\text{Proj} Q$
are unions of reduced linear spaces. Since the depths of $Q/a'$ and $Q/b'$ are positive, we
know that $a'$ and $b'$ are the saturated ideals of these reduced schemes and are thus radical
ideals which are intersections of primes generated by linear forms.

\[ \square \]

**Corollary 8.** Fix a power series $h(s) \in \mathbb{Z}[\![s]\!]$, and a non-negative integer $d$.
Consider the set $\Pi$ of pairs $(Q/a, k)$, where $a$ is a homogeneous ideal with $H_{Q/a}(s) = h(s)$
and $\text{depth} Q/a \geq d$, and $k = Q/(x_1, \ldots, x_n)$.

If $\Pi$ is non-empty, then there is a pair $(R', k) \in \Pi$ such that $P_{k}^{R}(s;t) \preceq P_{k}^{R'}(s;t)$ for
each $(R, k) \in \Pi$. We may take $R' = Q/a'$ where $a'$ is a $d$-lexicographic ideal. If $d > 0$
and $k$ is an infinite field, then we may instead take $R' = Q/c$ where $c$ is an intersection of
prime ideals generated by linear forms.

When $d = 0$, this is proven by Peeva \cite{24} and is also a special case of Theorem 1.
Proof. Let \( a' \) be the \( d \)-lexicographic ideal such that \( (Q/a', k) \in \Pi \), and let \( a \) be any ideal such that \( (Q/a, k) \in \Pi \). We must show that \( P_k^R(s; t) \preceq P_k^{R'}(s; t) \), where \( R = Q/a \) and \( R' = Q/a' \).

As in the proof of Theorem 1, we may assume that \( k \) is an infinite field and then reduce to the case that \( a \) is a Borel-fixed ideal such that \( x_{n-d+1}, \ldots, x_n \) is a regular sequence of linear forms on \( Q/a \). Then both \( a \) and \( a' \) are contained in the \( d \)-lexicographic ideal \( b = (x_1, \ldots, x_{n-d}) \). So if we let \( S = Q/b \), then Theorem 1 tells us that \( P_S^R(s; t) \preceq P_S^{R'}(s; t) \).

Now, \( x_{n-d+1}, \ldots, x_n \) is a regular sequence of linear forms on \( S \) while \( S/(x_{n-d+1}, \ldots, x_n) = k \). So,

\[
P_k^R(s; t) = (1 + st)^{n-d} P_S^R(s; t) \preceq (1 + st)^{n-d} P_S^{R'}(s; t) = P_k^{R'}(s; t).
\]

Now, we prove the last statement. Let \( a' \) be as above, and let \( c = \sigma_L(a'^{(p)}) \) as in the proof of Corollary 7. As in the proof of that corollary, we have that \( c \) is an intersection of prime ideals generated by linear forms. Since \( Q/\sigma_L(m^{(p)}) = Q/m = k \), we know that \( P_k^{Q/a'}(s; t) = P_k^{Q/c}(s; t) \).

**Theorem 2.** Fix a power series \( h(s) \in \mathbb{Z}[[s]] \), a non-negative integer \( d \), and a finitely generated graded free \( Q \)-module \( F \).

Let \( \Pi \) be the set of pairs \( (Q, F/M) \), where \( M \subseteq F \) is a homogeneous submodule such that \( H_{F/M}(s) = h(s) \) and \( \text{depth} F/M \geq d \).

If \( \Pi \) is non-empty, then there exists a pair \( (Q, F/M') \in \Pi \) such that \( P_{F/M'}^Q(s; t) \preceq P_{F/M'}^Q(s; t) \) for each \( (Q, F/M) \in \Pi \). We may take \( M' \) to be a \( d \)-lexicographic submodule of \( F \).

**Proof.** We may assume that \( k \) is an infinite field, as in the proof of Theorem 1. Likewise, we may assume that \( M \) is a Borel-fixed submodule of \( F \). Then, as in the proof of Theorem 1, we may reduce to the case in which \( d = 0 \). This case is Theorem 31 in [22].

Our proof of Theorem 2 is essentially the same as the proof of Elias’ theorem in the Cohen-Macaulay case [11].

### 4: Multiplicative Structures

This section discusses stable ideals and their minimal free resolutions as given by Eliahou and Kervaire [12], and the algebra structure on these resolutions constructed by Peeva [24].

Since the rings and modules in this section and the next are usually multigraded, we begin with a discussion on multigradings.

Let \( N^p = \oplus_{i=1}^p N \epsilon_i \). We will consider \( N^p \)-graded rings \( A \) such that \( A_0 = k \), a field. For example, the polynomial ring \( Q \) has a natural \( N^n \)-grading induced by \( \deg x_i = \epsilon_i \). If \( a \subseteq Q \) is a monomial ideal, then \( Q/a \) is also \( N^p \)-graded.

Let \( N^q = \oplus_{i=1}^q N \epsilon_i \) where \( q \geq p \), so that \( N^p \) is a summand of \( N^q \). For \( A \) as above, we consider \( N^q \)-graded modules \( M \). All ring and module homomorphisms will be homogeneous. If \( \dim_k M_\mu \) is finite for every \( \mu \in N^q \), then we write \( H_M(s) \) for the Hilbert series \( \sum_\mu \dim_k M_\mu s^\mu \in \mathbb{Z}[[s_1, \ldots, s_q]] \). If \( W = \bigoplus_{i \in \mathbb{N}} W_i \) where each \( W_i \) is an \( N^q \)-graded \( A \)-module, then we consider \( W \) as a \( N^{q+1} \)-graded \( A \)-module by letting an element of degree \( \mu \) in \( W_i \) have degree \( \mu + i \epsilon_{q+1} \) in \( W \). Thus, \( W \) has Hilbert series \( H_W(s; t) = \sum H_{W_i}(s) t^i \).
As in the usual graded case, a finitely generated \(\mathbb{N}^q\)-graded \(A\)-module \(M\) has a homogeneous minimal free resolution that is unique up to isomorphism. The number of elements of degree \(\mu\) in a minimal set of homogeneous generators of the \(i\)th free module in this resolution is the \((i, \mu)\) Betti number of \(M\), an invariant of \(M\) denoted \(\beta_{i\mu}^A(M)\). The multigraded Poincaré series of \(M\) as an \(A\)-module is \(P_M^A(s,t) = \sum \beta_{i\mu}^A(M) s^\mu t^i\). We recover the usual graded Poincaré series by setting \(s_1 = \cdots = s_q = s\) and the ungraded Poincaré series by setting \(s = 1\). Since \(M\) is \(\mathbb{N}^q\)-graded so is the vector space \(\text{Tor}_i^A(M,k)\) and \(\beta_{i\mu}^A(M) = \dim_k \text{Tor}_i^A(M,k)\). Thus, if we view \(\text{Tor}_i^A(M,k)\) as an \(\mathbb{N}^{q+1}\)-graded module, then \(P_M^A(s,t) = H_{\text{Tor}_i^A(M,k)}(s,t)\).

Let \(A\) be an \(\mathbb{N}^p\)-graded algebra, and let \(W\) be a complex of \(\mathbb{N}^p\)-graded \(A\)-modules
\[
\cdots \rightarrow W_2 \rightarrow W_1 \rightarrow W_0 \rightarrow 0.
\]

We write \(W^\# = \bigoplus W_i\) for the underlying \(\mathbb{N}^{p+1}\)-graded \(A\)-module. If \(w \in W_i\), we write \(|w| = i\).

In this paper, a differential graded (DG) algebra over an \(\mathbb{N}^p\)-graded algebra \(A\) is a non-negative complex \((X,\partial)\) of \(\mathbb{N}^p\)-graded \(A\)-modules with \(X^0\) an \(\mathbb{N}^{p+1}\)-graded \(A\)-algebra such that the Leibniz formula \(\partial(x_1x_2) = (\partial x_1)x_2 + (-1)^{|x_1|x_1}(\partial x_2)\) holds for all homogeneous \(x_1, x_2 \in X\). A homomorphism \(\varphi : (X,\partial) \rightarrow (X',\partial')\) of DG algebras is a homomorphism of the underlying graded algebras with \(\partial'\varphi = \varphi\partial\).

A DG module \(Y\) over a DG algebra \(X\) is a complex \((Y,\partial)\) of \(\mathbb{N}^p\)-graded \(A\)-modules with \(Y^0\) a \(\mathbb{N}^{p+1}\)-graded \(X^0\)-module such that \(\partial(xy) = (\partial x)y + (-1)^{|x|}x(\partial y)\) for all homogeneous \(x \in X\) and \(y \in Y\).

Let \(I \subset \mathbb{N}\) be a finite set. We denote by \(\max(I)\) the maximal element of \(I\) and by \(\min(I)\) the minimal element of \(I\). We denote by \(G(a)\) the minimal set of monomial generators of a monomial ideal \(a\). For a monomial \(a \in Q = k[x_1, \ldots, x_n]\), set \(\max(a) = \max(I)\), and \(\min(a) = \min(I)\), where \(I = \{i \mid x_i \text{ divides } a\}\).

An ideal \(a\) is said to be stable if it satisfies the following conditions:

(i) \(a\) is generated by monomials, and
(ii) If \(a \in a\), then \(\frac{x_i a}{\max(a)} \in a\) for all \(i \leq \max(a)\).

Note that \(d\)-lexicographic ideals are stable.

For each monomial \(a \in a\) there is a unique generator of \(a\), which we denote by \(\text{gen}_a(a)\), such that \(a = \text{gen}_a(a)a'\) with \(\max(\text{gen}_a(a)) \leq \min(a')\). For each monomial \(a \in a\), we define \(a_{(k)}\) to be \(\text{gen}_a(ax_k)\).

Next we describe the minimal \(Q\)-free resolution of a stable ideal, as constructed by Eliahou and Kervaire.

Let \(a\) be a stable ideal, and let \(E_k\) be the \(\mathbb{N}^{n+1}\)-graded \(Q\)-module underlying the exterior algebra
\[
\bigwedge \left( \bigoplus_{i=1}^n Qe_i \right) / (e_k, \ldots, e_n),
\]
with \(\deg e_i = \epsilon_i + \epsilon_{n+1}\). Consider the graded \(Q\)-module
\[
X = Q \oplus \left( \bigoplus_{a \in G(a)} Qa \otimes E_{\max(a)} \right),
\]
where \( \tilde{a} \) is a free generator of degree \( \deg(a) + \varepsilon_{n+1} \). For a finite subset \( I \subseteq \mathbb{N} \), we denote by \([a|I]\) the element \( \tilde{a} \otimes e_I \in X \). Eliahou and Kervaire show that \((X, \partial)\) with

\[
\partial[a|\emptyset] = a,
\]

\[
\partial[a|I] = \sum_{k=1}^{\text{card}(I)} (-1)^k x_{ik} [a|I_k] + \sum_{k=1}^{\text{card}(I)} (-1)^{k+1} \frac{x_{ik} a}{a_{(ik)}} [a_{(ik)}|I_k] \quad \text{where } I_k = I \setminus i_k,
\]

is a minimal \( \mathbb{N}^n \)-graded free resolution of \( Q/\mathfrak{a} \), of which the \( i \)-th free module is generated by the elements of degree \( \mu + i \varepsilon_{n+1} \) where \( \mu \in \mathbb{N}^n \). Thus, the multigraded Poincaré series of \( Q/\mathfrak{a} \) as a \( Q \)-module is

\[
P^Q_{Q/\mathfrak{a}}(s; t) = \sum_{c \in C} s^{\deg c} t^{\max(c) - 1} \prod_{i=1}^{\max(c) - 1} (1 + s_i t_i) \in \mathbb{Z}[s; t] = \mathbb{Z}[s_1, \ldots, s_n; t].
\]

Peeva constructed a DG algebra structure on \( X \) as follows:

For elements \( a, a' \in \mathfrak{a} \), define the *meet of \( a \) and \( a' \) to be* \( g = \text{gen}_a(\text{lcm}(a, a')) \). Suppose that \( x_{s_1} \cdots x_{s_f} = \frac{g}{\gcd(g, a)} \), with \( s_1 \leq \cdots \leq s_f \). Set \( a_0 = a \) and \( a_{i+1} = (a_i)_{(s_{i+1})} \), for \( 0 \leq i \leq f - 1 \). Similarly, if \( x_{t_1} \cdots x_{t_f'} = \frac{g}{\gcd(g, a')} \) with \( t_1 \leq \cdots \leq t_{f'} \), then we define a sequence with \( a'_0 = a' \) and \( a'_{j+1} = (a'_j)_{(t_{j+1})} \).

The integers \( s_1, \ldots, s_f \) (respectively, \( t_1, \ldots, t_f' \)) are called *\( a' \)-transforming* (respectively, *\( a' \a \)-transforming) elements. Note that \( a_f = g = b_{f'} \).

The product on basis elements \([a|I]\) and \([a'|J]\) in \( X \) is defined by the following formulas. In formula (2) we write \([a|I, s, J]\) for the image of \( \tilde{a} \otimes (e_I \wedge e_s \wedge e_J) \) in \( X \). In formulas (3.1-4) the \( s_i \) are the \( a' \)-transforming elements and the \( t_j \) are the \( a'a \)-transforming elements.

\[
\begin{align*}
(1) & \quad [a|I] \cdot [a|J] = 0 \\
(2) & \quad [a|I] \cdot [a(s)|J] = (-1)^{(\#I + 1)(\#J + 1)} [a|J] \cdot [a|I] \\
& \quad = \begin{cases} 0 & \text{if } s \leq \text{max}(J); \\
\frac{a_{(s)}}{x_s} [a|I, s, J] & \text{if } s > \text{max}(J). 
\end{cases} \\
(3.1) & \quad [a|I] \cdot [a'|J] = 0 \quad \text{if for some } i \text{ and } j, \text{ } s_i \in I \text{ and } t_j \in J. \\
(3.2) & \quad [a|I] \cdot [a'|J] = \sum_{i < z} \frac{aa'}{a_i a_{i+1}} [a_i|I] \cdot [a_{i+1}|J] \\
& \quad \quad \text{if } s_{z+1} \in I, s_i \notin I \text{ for } i \leq z, \text{ and } t_j \notin J \text{ for all } j. \\
(3.3) & \quad [a|I] \cdot [a'|J] = \sum_{j < y} \frac{aa'}{a_j a'_{j+1}} [a_j'|I] \cdot [a_j'|J] \\
& \quad \quad \text{if } t_{y+1} \in J, t_j \notin J \text{ for } j \leq y, \text{ and } s_i \notin I \text{ for all } i. \\
(3.4) & \quad [a|I] \cdot [a'|J] = \sum_i \frac{aa'}{a_i a_{i+1}} [a_i|I] \cdot [a_{i+1}|J] + \sum_j \frac{aa'}{a_j a'_{j+1}} [a_j'|I] \cdot [a_j'|J] \\
& \quad \quad \text{otherwise.}
\end{align*}
\]
This section is dedicated to proving the following statement:

**Theorem 9.** Let \( a \) and \( b \) be stable ideals such that \( a \subseteq b \cap (x_1, \ldots, x_n)^2 \), and consider \( S = Q/b \) as a module over \( R = Q/a \) via the canonical surjection.

The multigraded Poincaré series of \( S \) as an \( R \) module is given by

\[
P^R_S(s; t) = \frac{1 + h_{G(b)}(s; t) - (1 + t)h_{G(a) \cap G(b)}(s; t)}{1 - t \cdot h_{G(a)}(s; t)}.\]

Setting \( s_1 = \cdots = s_n = s \) yields Theorem 3.

The proof is given at the end of this section. This is an outline of the proof: for stable ideals \( a \subseteq b \), we establish that the minimal \( Q \)-free resolution of \( Q/b \) is a DG module over the DG algebra resolution of \( Q/a \). This allows us to use the degeneration of the Avramov spectral sequence to compute the Poincaré series of \( Q/b \) over \( Q/a \).

**Proposition 10.** Let \( a \) and \( b \) be stable ideals in \( Q \) with \( a \subseteq b \). Suppose that \( X \) is the minimal free resolution of \( Q/a \) and that \( Y \) is the minimal free resolution \( Q/b \), with DG algebra structures as in Section 4.

The \( k \)-linear map \( \phi: X \to Y \) defined by

\[
\phi([a|I]) = \frac{a}{\text{gen}_b(a)}[\text{gen}_b(a)|I]
\]

is a homomorphism of DG algebras, extending the canonical surjection \( Q/a \to Q/b \).

The proof of the proposition is based on a series of lemmas.

**Lemma 11.** If \( a \in b \), then \( \text{gen}_b(ax_i) = \text{gen}_b(\text{gen}_b(a)x_i) \) for all \( i \).

*Proof.* This is part of Lemma 1.3 in [12]. \( \square \)

**Lemma 12.** If \( a \in a \) and \( b = \text{gen}_b(a) \), then \( \text{gen}_b(a_{(i)}) = b_{(i)} \), where \( a_{(i)} = \text{gen}_a(ax_i) \) and \( b_{(i)} = \text{gen}_b(bx_i) \). In particular, \( \phi([a_{(i)}|I]) = \frac{a_{(i)}}{b_{(i)}}[b_{(i)}|I] \) for all \( i \).

*Proof.* Consider the series of equalities:

\[
\begin{align*}
\text{gen}_b(a_{(i)}) &= \text{gen}_b(a_{(i)} \cdot \frac{ax_i}{a_{(i)}}) \quad \text{as } \max(a_{(i)}) \leq \min(\frac{ax_i}{a_{(i)}}) \\
&= \text{gen}_b(ax_i) = \text{gen}_b(\text{gen}_b(a)x_i) \quad \text{by Lemma 11} \\
&= \text{gen}_b(bx_i) = b_{(i)}. 
\end{align*}
\]

This proves the first part of the lemma. Applying the definition of \( \phi \) in Proposition 10 gives the expression for \( \phi[a_{(i)}|I] \). \( \square \)

Next, we record some properties of transforming elements.
Lemma 13. Let \( g \) be the meet of elements \( a \) and \( a' \), and let \( s_1, \ldots, s_f \) be \( aa' \)-transforming elements.

1. \( x_{s_i} \) divides \( g \), in particular \( s_i \leq \max(g) \), for all \( i \leq f \).

2. Suppose that \( a <_{\text{lex}} a' \). Let \( i \) be the smallest index such that the power of \( x_i \) in \( a \) differs from the power of \( x_i \) in \( a' \). Then

\[
a = \alpha \beta \quad \text{with} \quad \max(\alpha) \leq i < \min(\beta),
\]
\[
a' = \alpha x_i^l \beta' \quad \text{for some} \ l \geq 1,
\]

and \( i \) is the first \( aa' \)-transforming element.

Proof. (1) is from the definition of the meet. The first part of (2) is from the definition of lexicographic order. The last statement is a straightforward computation using the definition of transforming elements.

Lemma 14. Let \( s_1, \ldots, s_f \) be \( aa' \)-transforming elements. Set \( b = \text{gen}_b(a) \) and \( b' = \text{gen}_b(a') \). Set \( b_i = \text{gen}_b(a_i) \) for \( 0 \leq i \leq f \). Similarly, for \( a'a' \)-transforming elements \( t_1, \ldots, t_{f'} \), set \( b'_j = \text{gen}_b(a'_j) \) for \( 0 \leq j \leq f' \).

There exist integers \( h \leq f \) and \( h' \leq f' \) with the following properties:

1. \( b_h = b_{h+1} = \cdots = b_f \) and \( b'_{h'} = b'_{h'+1} = \cdots = b'_{f'} \).

2. \( s_1, \ldots, s_h \) are \( bb' \)-transforming elements, and \( t_1, \ldots, t_{h'} \) are \( b'b' \)-transforming elements.

In particular, \( b_h = b'_{h'} \) is the meet of \( b \) and \( b' \), and \( \max(b_h) \leq s_{h+1} \) and \( \max(b'_{h'}) \leq t_{h'+1} \).

Proof. By symmetry, it is enough to show this for the \( aa' \)-transforming elements. To begin with, note that \( b_{i+1} = (b_i)(s_{i+1}) \) by Lemma 12.

Let \( h \) be the least integer such that \( b_h = b_{h+1} \). This is equivalent to \( \max(b_h) \leq s_{h+1} \).

As \( \max(b_{h+1}) \leq s_{h+1} \leq s_{h+2} \), we get \( b_{h+2} = (b_h)(s_{h+2}) = b_{h+1} \). An iteration yields \( b_h = b_{h+1} = \cdots = b_f \).

Suppose that \( b_{i-1} \neq b_i \); that is, \( s_i < \max(b_{i-1}) \). It is enough to show that \( s_i \) is the first \( b_{i-1}b' \)-transforming element. We prove this when \( a_{i-1} <_{\text{lex}} a' \); the proof is similar when \( a' <_{\text{lex}} a_{i-1} \). By Lemma 13, as \( s_i \) is the first \( a_{i-1}a' \)-transforming element, we have

\[
a_{i-1} = \alpha \beta \quad \text{with} \quad \max(\alpha) \leq s_i < \min(\beta),
\]
\[
a' = \alpha x_{s_i}^l \beta' \quad \text{for some} \ l \geq 1.
\]

Thus, \( s_i < \max(b_{i-1}) = \max(\text{gen}_b(a_{i-1})) \) which implies that \( \alpha \not\in b \). So, \( b_{i-1} = \alpha \gamma \) and \( b' = \alpha x_{s_i}^l \gamma' \), for some \( l' \geq 1 \). A further application of Lemma 13 completes the proof.

Proof of Proposition 10. To prove that \( \phi \) is a homomorphism of DG algebras we need to show that \( \phi \) is a morphism of complexes, and that it is a homomorphism of algebras.

Step 1. \( \phi \) is a morphism of complexes.

Let \( [a|I] \) be a basis element of \( X \), and let \( b = \text{gen}_b(a) \). The formulas for \( \partial \) and \( \phi \), along with Lemma 12, yield

\[
\phi \partial([a|I]) = \sum_{k \geq 1} (-1)^k \frac{\partial_{x_{ik}} a}{b} [b|I_k] + \sum_{k \geq 1} (-1)^k \frac{\partial_{x_{ik}} a}{b_{(ik)}} [b_{(ik)}|I_k] = \partial \phi([a|I]).
\]
This proves that $\phi$ is a morphism of complexes.

**Step 2. $\phi$ is a homomorphism of algebras.**

It is enough to prove that

$$
\phi([a|I] \cdot [a'|J]) = \phi([a|I]) \cdot (\phi[a'|J])
$$

for basis elements $[a|I]$ and $[a'|J]$ of $X$. There are three cases to consider, depending on which of the multiplication laws at the end of Section 4 is applied to compute the product of $[a|I]$ and $[a'|J]$.

**Type 1.** If $a = a'$, then $\phi([a|I] \cdot [a'|J]) = \phi([a|I])\phi([a'|J]) = 0$, by multiplication law (1).

**Type 2.** Suppose that $a' = a_{(s)}$ for some integer $s$. Set $b = \phi(a)$. Using Lemma 12, we get $\phi([a|I]) \cdot \phi([a_{(s)}|J]) = \frac{a_{a_{(s)}}}{b_{b_{(s)}}}[b[I] \cdot [b(s)|J]$. If $s \leq \max(J)$, then $\phi([a|I] \cdot [a'|J]) = \phi([a|I]) \cdot \phi([a|J]) = 0$, by multiplication law (2). If $s > \max(J)$, then a direct computation shows that

$$
\phi([a|I] \cdot [a_{(s)}|J]) = \phi}\left(\frac{a_{a_{(s)}}}{x_s}[a[I, s, J]\right) = \frac{a_{a_{(s)}}}{x_s}b[I, s, J] = \frac{aa_{(s)}}{bb_{(s)}}[b[I] \cdot [b(s)|J].
$$

This completes the proof for this case.

**Type 3.** There are four cases to consider. Let the notation be as in Lemma 14.

**Case (i).** Suppose that $s_z \in I$ and $t_y \in J$. As $[a|I] \cdot [a'|J] = 0$ by multiplication law (3.1), we must show that $[b[I] \cdot [b'|J] = 0$.

If $z \leq h$ and $y \leq h'$, then $s_z$ is a $bb^r$-transforming element and $t_y$ is a $b^r b$-transforming element by Lemma 14. We apply multiplication law (3.1) to see that $[b[I] \cdot [b'|J] = 0$.

If $z > h$, then $b_z = b^r_h$ is the meet of $b$ and $b'$, and max($b_z$) $\leq s_z$ by Lemma 14. As $t_1, \ldots, t_h$ are $b' b$-transforming elements, cf. Lemma 14, it follows from Lemma 13 that $t_j \leq \max(b_z) \leq s_z$, for all $1 \leq j \leq h'$. Note that $s_z \in I$; thus, max($I$) $\geq t_j$ and so $[b[I] \cdot [b'|J] = 0$ for all $1 \leq j \leq h'$, by multiplication law (2).

An analogous argument shows that if $y > h'$ then $[b_{t_{i-1}}[I] \cdot [b_i|J] = 0$ for all $0 \leq i \leq h$.

Now, applying the appropriate case of multiplication law (3), we deduce that $[b[I] \cdot [b'|J] = 0$.

**Case (iv).** Suppose that $s_i \not\in I$ and $t_j \not\in J$ for all $i, j$. In the series of equalities below, the first is by multiplication law (3.4), the second is by the result of Type 2 above, and the last equality follows from the fact that $b_i = b_{i+1}$ and $b'_j = b'_{j+1}$, for $i \geq h$ and $j \geq h'$, by Lemma 14.

$$
\phi([a[I] \cdot [a'|J]) = \sum_{0 \leq i < f} \frac{aa'}{a_i a_{i+1} b_i} [a[I] \cdot [a_{i+1}] \cdot J] + \sum_{0 \leq j < f'} \frac{aa'}{a_j a_{j+1} b'_j} [a[J] \cdot [a'_j] \cdot J])
$$

$$
= \sum_{0 \leq i < h} \frac{aa'}{b_i b_{i+1}} [b[I] \cdot [b_{i+1}] \cdot J] + \sum_{0 \leq j < h'} \frac{aa'}{b'_j b'_j} [b'[J] \cdot [b'_j] \cdot J]
$$

$$
= \sum_{0 \leq i < h} \frac{aa'}{b_i b_{i+1}} [b[I] \cdot [b_{i+1}] \cdot J] + \sum_{0 \leq j < h'} \frac{aa'}{b'_j b'_j} [b'[J] \cdot [b'_j] \cdot J].
$$
Note that $s_1, \ldots, s_f$ (respectively, $t_1, \ldots, t_{f'}$) are $bb'$ (respectively $b'b$) transforming elements, by Lemma 14. As $s_i \not\in I$ and $t_j \not\in J$, applying multiplication law (3.4) yields:

$$
\phi([a|I]) \cdot \phi([a'|J]) = \frac{aa'}{bb'} [b|I] \cdot [b'|J] \\
= \sum_{0 \leq i < h} \frac{aa'}{b_i b_{i+1}} [b_i|I] \cdot [b_{i+1}|J] + \sum_{0 \leq j < h'} \frac{aa'}{b_j' b_{j+1}'} [b_j'|I] \cdot [b_{j+1}'|J] \\
= \phi([a|I] \cdot [a'|J]).
$$

The proofs for cases (ii) and (iii) are similar to that for case (iv).

Now we prove Theorem 9. The technique is similar to the proof of the theorem in Section 2.1 of [5].

**Proof of Theorem 9.** Let $A = \text{Tor}^Q(R, k)$ and $B = \text{Tor}^Q(S, k)$. Then $A$ is an $\mathbb{N}^{n+1}$-graded ring with $A_0 = k$ and $B$ is an $\mathbb{N}^{n+1}$-graded $A$-module by the homology products (cf. Chapter VIII of [20]).

Let $X$ and $Y$ be the Eliahou-Kervaire resolutions of $R$ and $S$ respectively, with DG algebra structures as in Section 5. By Proposition 10, $Y$ is a $\mathbb{N}^{n+1}$-graded DG module over $X$ via the homomorphism $\phi$. Notice that as $\mathbb{N}^{n+1}$-graded $Q$-modules, $A = X \otimes_Q k$ and $B = Y \otimes_Q k$. In fact, the homomorphism $\phi \otimes_Q k : A \to B$ coincides with the action of $A$ on $B$ given above.

The Avramov spectral sequence (Theorem 3.1 of [4] or Proposition 3.2.4 of [6]),

$$
E_{p,q}^2 = \text{Tor}_p^A(B, k)_q \Longrightarrow \text{Tor}_{p+q}^R(S, k)
$$

is compatible with the $\mathbb{N}^n$-grading. (Here, $\text{Tor}_p^A(B, k)_q$ is the $\mathbb{N}^n$-graded $Q$-module generated by elements of degree $\mu + q\epsilon_{n+1}$ in $\text{Tor}_p^A(B, k)$ such that $\mu \in \mathbb{N}^n$.) Since $X$ and $Y$ are minimal and $Y$ is a DG module over $X$, the sequence collapses with $E_{p,q}^2 = E_{p,q}^\infty$ by Proposition 4.1.1 of [4]. This implies an equality of formal power series:

$$
P_S^R(s; t) = P_B^A(s; t; t).
$$

Set $U = \text{Im}(\phi \otimes k) \cong (A/Z)$, where $Z = \text{Ker}(\phi \otimes k)$, and let $V$ be an $\mathbb{N}^{n+1}$-graded subspace of $A$, complementary to $U$. Peeva’s multiplication laws, given at the end of Section 5, and Proposition 10, together with the assumption that $a \subseteq (x_1, \ldots, x_n)^2 \cap b$, show that

$$
A_+ \cdot A_+ = 0, \quad A_+ \cdot B_+ = 0, \quad A_+ \cdot B \subset U,
$$

where $A_+$ and $B_+$ are the homogeneous maximal ideals of $A$ and $B$. Thus, $B = U \oplus V$ as graded $A$-modules, with $A_+ \cdot V = 0$. This last equality implies that as an $A$-module, $V$ is a direct sum of copies of $k = A/A_+$ with generators in various degrees. In terms of Poincaré series this is tantamount to

$$
P_B^A(s; t; u) = P_U^A(s; t; u) + P_k^A(s; t; u) \cdot H_V(s; t).
$$
The short exact sequence $0 \to Z \to A \xrightarrow{φ} U \to 0$ yields isomorphisms $\text{Tor}^A_i(U, k) = \text{Tor}^A_{i-1}(Z, k)$, for $i \geq 1$. This, with $A_+ \cdot A_+ = 0$, gives

$$P^A_U(s; t; u) = 1 + uP^A_Z(s; t; u) = 1 + uP^A_k(s; t; u) \cdot H_Z(s; t).$$

Thus, $P^A_B(s; t; u) = 1 + uP^A_k(s; t; u)H_Z(s; t) + P^A(s; t; u) \cdot H_V(s; t)$.

Using the short exact sequence $0 \to A_+ \to A \to k \to 0$, and $A_+ \cdot A_+ = 0$, we deduce the well known recurrence relation:

$$P^A_k(s; t; u) = 1 + uP^A_k(s; t; u) \cdot H_{A_+}(s; t).$$

This shows that $P^A_k(s; t; u) = (1 - uh_G(a)(s; t))^{-1}$, since $H_{A_+} = h_G(a)$.

It remains to compute $H_V(s; t)$ and $H_Z(s; t)$. Note that $U$ is the subspace of $B$ spanned by $B_0 = k$ and basis elements $\{[c|J]|c \in G(a) \cap G(b)\}$, and that $Z$ is the subspace of $A$ spanned by elements $\{(a|I)|a \in G(a) \setminus G(b)\}$. Hence,

$$H_U(s; t) = 1 + h_{G(a) \cap G(b)}(s; t),$$

$$H_V(s; t) = 1 + h_{G(b)}(s; t) - H_U(s; t),$$

$$H_Z(s; t) = 1 + h_{G(a)}(s; t) - H_U(s; t).$$

Putting all this together, and setting $u = t$, yields the required expression for the Poincaré series. □

6: Local Rings

In this section, we reformulate theorems 1, 2 and 3 and Corollary 8 for local rings. If $A$ is a regular local ring with maximal ideal $m$ with a fixed set of minimal generators $x_1, \ldots, x_n$ of $m$, and $F$ is a free $A$-module (such as $A$ itself) with a fixed basis $f_1, \ldots, f_r$, then one can define monomials of $F$, monomial submodules of $F$, and lexicographic submodules of $F$; cf. Section 7 of [22] for these definitions. As in the graded case, we define a $d$-lexicographic submodule to be a monomial submodule $L \subseteq F$ such that $x_{n-d+1}, \ldots, x_n$ is a regular sequence on $F/L$ and the image of $L$ in the free $A/(x_{n-d+1}, \ldots, x_n)$-module $F/(x_{n-d+1}, \ldots, x_n)F$ is a lexicographic $A/(x_{n-d+1}, \ldots, x_n)$-submodule. If $B$ is a local noetherian ring with maximal ideal $m$ and $M$ is a $B$-module, then the Hilbert series $H_M(s)$ is the Hilbert series of the associated graded module $\text{gr}_MM$.

Lemma 15. Let $A$ be a regular local ring with maximal ideal $m$. If $I \subseteq J$ are monomial ideals in $A$, $B = A/I$ and $C = A/J$, then $P^B_C(t) = P^B_{\text{gr}_mB}(t)$. If $F$ is a free $A$-module and $M \subseteq F$ is a monomial submodule then $P^A_{F/M}(t) = P^A_{\text{gr}_m(F/M)}(t)$.

Proof. The second statement is part of Lemma 32 in [22], and the proof of the first statement is a straightforward generalization of the proof of that lemma. □
Lemma 16. Let \( h(s) \in \mathbb{Z}((s)) \) be a Laurent series and \( d \) a non-negative integer. There is a submodule \( M \subseteq F \) such that \( F/M \) has Hilbert series \( h(s) \) and \( F/M \) has depth at least \( d \) if and only if there is a \( d \)-lexicographic submodule \( L \) of \( F \) such that \( F/L \) has Hilbert series \( h(s) \).

Proof. If \( h(s) \) is the Hilbert series of a quotient module of \( F \) with depth at least \( d \), then it is also the Hilbert series of a graded quotient module of \( \text{gr}_m F \) with depth at least \( d \). Then, by Proposition 4, there is a \( d \)-lexicographic submodule \( L \) of \( \text{gr}_m F \) such that \( (\text{gr}_m F)/L \) has Hilbert series \( h(s) \). Let \( L \) be the submodule of \( F \) generated by the monomials in \( F \) corresponding to the monomial generators of \( L \) in \( \text{gr}_m F \). This is the \( d \)-lexicographic submodule that we seek. \( \square \)

Remark 17.

We recall the construction of a spectral sequence due to Serre, cf. Chapter II, Section A in [26]. Let \((B, m, k)\) be a noetherian local ring, and let \( M \) and \( N \) be finitely generated \( B \)-modules. There is a convergent spectral sequence residing in the first quadrant, with

\[
E^1_{m,q} = \text{Tor}^B_{m,q}(\text{gr}_m(M), \text{gr}_m(N)) \quad \quad \partial^r_{m,q} : E^r_{m,q} \to E^r_{m-1,q+r} \quad \text{for } r \geq 1
\]

and such that \( E^{\infty,*} = \bigoplus_{q \in \mathbb{Z}} E^{\infty}_{m,q} \) is the graded object associated to \( \text{Tor}^B_m(M, N) \) by an \( m \)-stable filtration. The sequence is constructed as follows: \( M \) has a \( B \)-free resolution

\[
X : \cdots \to X_{m+1} \xrightarrow{\partial_{m+1}} X_m \xrightarrow{\partial_m} X_{m-1} \to \cdots
\]
endowed with a decreasing filtration \( X = X(0) \supseteq X(1) \supseteq \cdots \) with \( \partial(X(q)) = (\partial X) \cap X(q) \), such that the graded object associated to \( X \) by this filtration is a homogeneous \( \text{gr}_m(B) \)-free resolution of \( \text{gr}_m(M) \). The complex \( Y = X \otimes_B N \) has the induced filtration \( Y(q) = \text{Im}(X(q) \otimes_B N \to X \otimes_B N) \), and the resulting spectral sequence is the one above. The filtration on \( H(Y) = \text{Tor}^B(M, N) \) is defined by the submodules \( \text{Im}(H(Y(q)) \to H(Y)) \).

In the special case when \( N = k \), one has \( E^1_{m,*} = \text{Tor}^B_m(\text{gr}_m(M), k) \), and from this it follows that \( P^B_M(t) \preceq P^{\text{gr}_m(B)}_{\text{gr}_m(M)}(t) \).

Theorem 18. Fix power series \( h_1(s), h_2(s) \in \mathbb{Z}[[s]] \) and a non-negative integer \( d \).

Let \( \Pi \) be the set of pairs \((B, C)\) where \( B \) is a local ring with \( H_B(s) = h_1(s) \) and \( \text{depth} B \geq d \), and \( C \) is a homomorphic image of \( B \), with \( H_C(s) = h_2(s) \) and \( \text{depth} C \geq d \).

If \( \Pi \) is non-empty, then there is a pair \((B', C')\) \( \in \Pi \) such that \( P^B_C(t) \preceq P^{B'}_{C'}(t) \) for each \((B, C) \in \Pi \).

Proof. Let \( m_B \) be the maximal ideal of \( B \). We may replace \( B \) and \( C \) by their \( m_B \)-adic completions without changing their Hilbert series, their depths, or the Poincaré series \( P^B_C(t) \). The structure theory of complete local rings tells us that there is a regular local ring \( A \) such that \( B = A/I \) and \( C = A/J \) with \( I \subseteq J \). Fix a minimal set of generators \( x_1, \ldots, x_n \) for the maximal ideal \( m_A \) of \( A \).

Consider the associated graded rings \( Q = \text{gr} A \), \( R = \text{gr} B \), and \( S = \text{gr} C \) with respect to the \( m_A \)-adic filtration. By Remark 17, \( P^B_C(t) \preceq P^R_S(t) \). So, with \( R' \) and \( S' \) as in Theorem 1, \( P^B_C(t) \preceq P^R_S(t) \).
By Lemma 16 there are $d$-lexicographic ideals $I' \subseteq J'$ such that $B' = A/I'$ and $C' = A/J'$ have Hilbert series $h_1(s)$ and $h_2(s)$ respectively. Then, $\text{gr}B' = R'$ and $\text{gr}C' = S'$ and

$$P_C^B(t) \preceq P_{S'}^R(t) = P_{C'}^{B'}(t)$$

by Lemma 15.

**Theorem 19.** Fix a power series $h(s) \in \mathbb{Z}[[s]]$ and a non-negative integer $d$.

Let $\Pi$ be the set of pairs $(B, k)$ where $B$ is a local ring such that $H_B(s) = h(s)$ and $\text{depth} B \geq d$.

If $\Pi \neq \emptyset$, then there is a pair $(B', k) \in \Pi$ such that $P_k^B(t) \preceq P_k^{B'}(t)$ for each $(B, k) \in \Pi$.

*Proof.* The proof is similar to the proof of Theorem 18, except that the problem is reduced to Corollary 8, rather than to Theorem 1. $\Box$

**Theorem 20.** Let $A$ be a regular local ring and $F$ be a finitely generated free $A$-module. Fix a Laurent series $h(s) \in \mathbb{Z}((s))$ and a non-negative integer $d$.

Let $\Pi$ be the set of pairs $(A, F/M)$ where $M \subseteq F$ is a homogeneous submodule such that $H_{F/M}(s) = h(s)$ and $\text{depth} F/M \geq d$.

If $\Pi$ is non-empty, then there exists a pair $(A, F/M') \in \Pi$ such that $P_k^A(t) \preceq P_k^{A'}(t)$, for each $(A, F/M) \in \Pi$. We may take $M'$ to be a $d$-lexicographic submodule of $F$.

*Proof.* The proof is similar to the proof of Theorem 18, except that it is not necessary to complete $A$ and $F$, and the problem is reduced to Theorem 2, rather than to Theorem 1. $\Box$

The following statement gives the maximal Poincaré series appearing in Theorem 18. Its proof is immediate from Theorem 3 and Lemma 15. Stable ideals in a regular local ring are defined as in Section 4 for a polynomial ring.

**Theorem 21.** Let $A$ be a regular local ring and let $x_1, \ldots, x_n$ be generators of the maximal ideal $m$. Let $I$ and $J$ be stable ideals generated by monomials in $x_1, \ldots, x_n$ such that $I \subseteq J \cap m^2$.

The Poincaré series of $A/J$ as an $A/I$-module is given by

$$P_{A/J}^A(t) = \frac{1 + t^2 \sum_{a \in G(I)} (1 + t)^{\max(a) - 1}}{1 - t^2 \sum_{a \in G(I)} (1 + t)^{\max(a) - 1} - \sum_{b \in G(J)} (1 + t)^{\max(b)}},$$

where $G(a)$ and $G(b)$ are minimal sets of monomial generators of $a$ and $b$ respectively.

7: Gorenstein Rings

A special case of both Theorem 1 and Theorem 2 is that if $\Pi$ consists of pairs $(Q, Q/b)$ such that $Q/b$ is Cohen-Macaulay with a given Hilbert series $h(s)$, then there is an element of $\Pi$ with a maximal Poincaré series. Likewise, a special case of Corollary 7 is that if $\Pi$ consists of pairs $(Q/a, k)$ such that $Q/a$ is Cohen-Macaulay with a given Hilbert series $h(s)$, then there is an element of $\Pi$ with a maximal Poincaré series. These cases suggest two questions.
**Question 22.** Let \( h(s) \in \mathbb{Z}[[s]] \) be the Hilbert series of a graded Gorenstein quotient ring of \( Q \). Is there a graded Gorenstein quotient ring \( R' \) of \( Q \) with Hilbert series \( h(s) \) such that \( P_R^Q(s;t) \preceq P_{R'}^Q(s;t) \) for every graded Gorenstein quotient ring \( R \) of \( Q \) with Hilbert series \( h(s) \)?

If \( \dim R \geq \dim Q - 2 \), then \( R = Q/I \) where \( I \) is generated by a regular sequence of degrees that can be determined from \( h(s) \). Thus, the Poincaré series of all such \( R \) are the same, so that the question becomes trivial. If \( \dim R = \dim Q - 3 \), then Diesel proved that the answer to the question is yes [9]. The question is open when \( \dim R < \dim Q - 3 \). When \( \dim R = \dim Q - 4 \), Geramita, Ko and Shin have conjectured that the answer is yes and have given a candidate for \( R' \) [15].

The technique used to prove Theorems 1 and 2 does not help us with this question since it is not always possible to deform a Gorenstein ideal to a monomial Gorenstein ideal. Indeed, in the Artinian case the only monomial Gorenstein ideals are complete intersections, while there are many Hilbert series that arise for Artinian Gorenstein rings that are not the Hilbert series of any complete intersection.

**Question 23.** Let \( h(s) \in \mathbb{Z}[[s]] \) be the Hilbert series of a graded Gorenstein quotient ring of \( Q \). Is there a graded Gorenstein quotient ring \( R' \) of \( Q \) with Hilbert series \( h(s) \) such that \( P_k^R(s;t) \preceq P_k^{R'}(s;t) \) for every graded Gorenstein quotient ring \( R \) of \( Q \) with Hilbert series \( h(s) \)?

If \( \dim R \geq \dim Q - 2 \), than \( R \) is a complete intersection as above. In this case the Poincaré series of all such \( R \) are the same as well. See [27] for these series. If \( \dim R < \dim Q - 2 \), then the question is open.

For other theorems and conjectures on maximal Poincaré series, see [2] and [3].

**References**


H. Hulett, Maximum Betti numbers of homogeneous ideals with a given Hilbert function, Communications in Algebra 21 (1993), 2335–2350.
J. Tate, Homology of Noetherian rings and local rings, Ill. J. of Math. 1 (1957), 14–27.

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