Sample Problem 1

Let \( x = x(t) \) be the position at time \( t \) of a particle of mass \( m \) moving under the influence of a force \( F \) be a force with potential \( f \). The total energy is given by the Hamiltonian function

\[
H(x, \dot{x}) = \frac{1}{2}m\|\dot{x}\|^2 + f(x).
\]

Show that the total energy is conserved.

Solution

The time derivative \( H \) is

\[
\frac{d}{dt}H(x, \dot{x}) = \frac{d}{dt} \left[ \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + f(x_1, x_2, x_3) \right]
\]

\[
= m(\ddot{x}_1 \dot{x}_1 + \ddot{x}_2 \dot{x}_2 + \ddot{x}_3 \dot{x}_3) + \frac{\partial f}{\partial x_1} \dot{x}_1 + \frac{\partial f}{\partial x_2} \dot{x}_2 + \frac{\partial f}{\partial x_3} \dot{x}_3
\]

\[
= m\ddot{x} \cdot \dot{x} + \nabla f \cdot \dot{x}
\]

\[
= F \cdot \dot{x} - F \cdot \dot{x}
\]

\[
= 0.
\]

Hence the energy remains constant, i.e. it is conserved.

A Few Points

1. Write in complete sentences. Capitalize and punctuate correctly. Don’t abbreviate arbitrarily. Don’t substitute symbols for words except in mathematical expressions. For example, you may use an arrow in the expression “\( x \rightarrow 0 \)” but not as a substitute for words like “therefore” and “hence.”

2. Use correct, consistent notation and terminology. Standard notation needn’t be explained. Otherwise, symbols should be defined.

3. Where necessary, provide the reader with a modicum of clear commentary to justify and motivate your mathematical calculations. Don’t write more than is necessary. Routine calculations should remain on the scratch paper. Simple steps shouldn’t be explained. For example, it is not necessary to tell me that you used the chain rule to get from (1) to (2), or that you used the fact that \( \nabla f = -F \) to get from (3) to (4). You could if you thought it necessary, but I’d prefer that you erred on the side of brevity.

4. Most of us can’t simply put pen to paper and spontaneously produce well-written solutions. You have to give some thought to the layout and the editing of the material. You might have to write several drafts.