The Inverse Laplace Transform

1. If $\mathcal{L}\{f(t)\} = F(s)$, then the inverse Laplace transform of $F(s)$ is

\[
\mathcal{L}^{-1}\{F(s)\} = f(t).
\]  
(1)

The inverse transform $\mathcal{L}^{-1}$ is a linear operator:

\[
\mathcal{L}^{-1}\{F(s) + G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\},
\]

and

\[
\mathcal{L}^{-1}\{cF(s)\} = c\mathcal{L}^{-1}\{F(s)\},
\]

for any constant $c$.

2. Example: The inverse Laplace transform of

\[
U(s) = \frac{1}{s^3} + \frac{6}{s^2 + 4},
\]

is

\[
u(t) = \mathcal{L}^{-1}\{U(s)\} = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + 3 \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}
\]

\[= \frac{s^2}{2} + 3 \sin 2t. \quad (4)
\]

3. Example: Suppose you want to find the inverse Laplace transform $x(t)$ of

\[
X(s) = \frac{1}{(s + 1)^4} + \frac{s - 3}{(s - 3)^2 + 6}.
\]

Just use the shift property (paragraph 11 from the previous set of notes):

\[
x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{s - 3}{(s - 3)^2 + 6}\right\}
\]

\[= \frac{e^{-t} t^3}{6} + e^{3t} \cos \sqrt{6}t.
\]

4. Example: Let $y(t)$ be the inverse Laplace transform of

\[
Y(s) = \frac{e^{-3s}}{s^2 + 4}.
\]
Don’t worry about the exponential term. Since the inverse transform of $s/(s^2 + 4)$ is $\cos 2t$, we have by the switchig property (paragraph 12 from the previous notes):

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s^2 + 4} \right\} = H(t - 3) \cos 2(t - 3).$$

5. Example: Let $G(s) = s(s^2 + 4) - 1$. The inverse transform of $G(s)$ is

$$g(t) = \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4s + 5} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{(s + 2)^2 + 1} \right\}
= \mathcal{L}^{-1} \left\{ \frac{s + 2}{(s + 2)^2 + 1} \right\} - \mathcal{L}^{-1} \left\{ \frac{2}{(s + 2)^2 + 1} \right\}
= e^{-2t} \cos t - 2e^{-2t} \sin t. \quad (5)$$

6. There is usually more than one way to invert the Laplace transform. For example, let $F(s) = (s^2 + 4s)^{-1}$. You could compute the inverse transform of this function by completing the square:

$$f(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4s} \right\}
= \mathcal{L}^{-1} \left\{ \frac{1}{(s + 2)^2 - 4} \right\}
= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{(s + 2)^2 - 4} \right\}
= \frac{1}{2} e^{-2t} \sinh 2t. \quad (6)$$

You could also use the partial fraction decomposition (PFD) of $F(s)$:

$$F(s) = \frac{1}{s(s + 4)} = \frac{1}{4s} - \frac{1}{4(s + 4)},$$

Therefore,

$$f(t) = \mathcal{L}^{-1} \{ F(s) \}
= \mathcal{L}^{-1} \left\{ \frac{1}{4s} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{4(s + 4)} \right\}
= \frac{1}{4} - \frac{1}{4} e^{-4t}
= \frac{1}{2} e^{-2t} \sinh 2t. \quad (7)$$
7. **Example:** Compute the inverse Laplace transform \( q(t) \) of

\[
Q(s) = \frac{3s}{(s^2 + 1)^2}.
\]

You could compute \( q(t) \) by partial fractions, but there's a less tedious way. Note that

\[
Q(s) = -\frac{3}{2} \frac{d}{ds} \frac{1}{s^2 + 1}.
\]

Hence,

\[
q(t) = \mathcal{L}^{-1} \{Q(s)\} = -\frac{3}{2} \mathcal{L}^{-1} \left\{ \frac{d}{ds} \frac{1}{s^2 + 1} \right\} = \frac{3}{2} t \sin t. \tag{8}
\]

8. **Definition:** The **convolution** of functions \( f(t) \) and \( g(t) \) is

\[
(f * g)(t) = \int_0^t f(t - v)g(v) \, dv. \tag{9}
\]

As we showed in class, the convolution is commutative:

\[
(f * g)(t) = \int_0^t f(t - v)g(v) \, dv = \int_0^t g(t - v)f(v) \, dv = (g * f)(t). \tag{10}
\]

9. **Example:** Let \( f(t) = t \) and \( g(t) = e^t \). The convolution of \( f \) and \( g \) is

\[
(f * g)(t) = \int_0^t (t - v)e^v \, dv = t \int_0^t e^v \, dv - \int_0^t ve^v \, dv = e^t - t - 1. \tag{11}
\]

10. **Proposition:** (The Convolution Theorem) If the Laplace transforms of \( f(t) \) and \( g(t) \) are \( F(s) \) and \( G(s) \) respectively, then

\[
\mathcal{L} \{ (f * g)(t) \} = F(s)G(s), \tag{12}
\]
that is,

\[ \mathcal{L}^{-1} \{ F(s)G(s) \} = (f * g)(t). \tag{13} \]

11. Suppose that you want to find the inverse transform \( x(t) \) of \( X(s) \). If you can write \( X(s) \) as a product \( F(s)G(s) \) where \( f(t) \) and \( g(t) \) are known, then by the above result, \( x(t) = (f * g)(t) \).

12. **Example:** Consider the previous example: Find the inverse transform \( q(s) \) of

\[ Q(s) = \frac{3s}{(s^2 + 1)^2}. \]

Write \( Q(s) = F(s)G(s) \), where

\[ F(s) = \frac{3}{s^2 + 1}, \]

and

\[ G(s) = \frac{s}{s^2 + 1}. \]

The inverse transforms are of \( F(s) \) and \( G(s) \) are \( f(t) = 3 \sin t \) and \( g(t) = \cos t \). Therefore

\[
q(s) = \mathcal{L}^{-1} \{ Q(s) \} \\
= \mathcal{L}^{-1} \{ F(s)G(s) \} \\
= (f * g)(t) \\
= 3 \int_0^t \sin (t - v) \cos v \, dv. \tag{14}
\]

Even if you stop here, you at least have a fairly simple, compact expression for \( q(s) \). To do the integral (14), use the trigonometric identity

\[ \sin A \cos B = \frac{\sin (A + B) + \sin (A - B)}{2}. \]

With this, (14) becomes

\[
q(s) = \frac{3}{2} \int_0^t \sin t \, dv + \int_0^t \sin (t - 2v) \, dv \\
= \frac{3}{2} t \sin t. \tag{15}
\]

13. **Example:** Find the inverse Laplace transform \( x(t) \) of the function

\[ X(s) = \frac{1}{s(s^2 + 4)}. \]
If you want to use the convolution theorem, write $X(s)$ as a product:

$$X(s) = \frac{1}{s} \frac{1}{s^2 + 4}.$$ 

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

and

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 4} \right\} = \frac{1}{2} \sin 2t,$$

we have

$$x(t) = \frac{1}{2} \int_0^t \sin 2v \, dv$$

$$= \frac{1}{4} (1 - \cos 2t).$$

You could also use the PFD:

$$X(s) = \frac{1}{4s} - \frac{s}{4(s^2 + 4)}.$$ 

Therefore,

$$x(t) = \mathcal{L}^{-1}\left\{ \frac{1}{4s} \right\} - \mathcal{L}^{-1}\left\{ \frac{s}{4(s^2 + 4)} \right\}$$

$$= \frac{1}{4} (1 - \cos 2t).$$