1 Abstract Measure Theory

Def 12: 1 (a) A collection $\mathcal{M}$ of subsets of $X$ is a $\sigma$-algebra if it has the following properties: i) $\emptyset, X \in \mathcal{M}$, ii) If $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$, and iii) If $\{E_i\}_{i=1}^\infty \subseteq \mathcal{M}$ then $\bigcup_{i=1}^\infty E_i \in \mathcal{M}$. (b) If $\mathcal{M}$ is a $\sigma$-algebra on $X$, then the pair $(X, \mathcal{M})$ is called a measurable space. If $\mathcal{M}$ is understood, then $X$ will be called a measurable space. The members of $\mathcal{M}$ are called measurable sets. (c) If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces and $f : X \to Y$ satisfies $f^{-1}(E) \in \mathcal{M}$ for each $E \in \mathcal{N}$, then $f$ is called $(\mathcal{M}, \mathcal{N})$-measurable. If $\mathcal{M}$ and $\mathcal{N}$ are understood, we just say $f$ is measurable.

Example 13: (a) If $X \neq \emptyset$, then $\mathcal{M} = \{\emptyset, X\}$ is the trivial measurable space. (b) If $X \neq \emptyset$ then $\mathcal{M} = \mathcal{P}(X)$ is the discrete measurable space. (c) If $X = \mathbb{R}$, then $\mathcal{M} = \{\emptyset, \{0\}, \mathbb{R} \setminus \{0\}, \mathbb{R}\}$ is a $\sigma$-algebra. (e) If $X \neq \emptyset$, then $\mathcal{M} = \{E \subseteq X : E \text{ or } E^c \text{ is countable}\}$ is a $\sigma$-algebra.

LMA 14: (Disjoint Dissection) If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{P}(X)$ then $\{F_k\}_{k=1}^\infty \subseteq \mathcal{P}(X)$ defined by $F_1 := E_1$ and $F_k = E_k \setminus (\bigcup_{j=1}^{k-1} E_j)$ for $k > 1$ is a sequence of mutually disjoint sets such that $\bigcup_{k=1}^\infty F_k = \bigcup_{j=1}^\infty E_j$.

Thm 15: 2 If $\mathcal{E}$ is a collection of subsets of $X$ then there is a smallest $\sigma$-algebra (which is unique) $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$.

Def 16: The unique smallest $\sigma$-algebra containing a subset $\mathcal{E} \subseteq \mathcal{P}(X)$ is called the $\sigma$-algebra generated by $\mathcal{E}$, and we denote this $\sigma$-algebra by $\mathcal{M}(\mathcal{E})$.

Def 17: Let $(X, T)$ be a topological space. The $\sigma$-algebra $\mathcal{M}(T)$ is called the Borel $\sigma$-algebra on $X$. It is denoted by $\mathcal{B}(X, T)$ or $\mathcal{B} X$. Elements of $\mathcal{B} X$ are called Borel sets.

Def 18: Let $(X, T)$ be a topological space. Set $F_\sigma := \{\text{countable unions of closed sets in } X\}$, $G_\delta := \{\text{countable intersections of open sets in } X\}$, $F_{\sigma\delta} := \{\text{countable intersections of sets in } F_\sigma\}$, etc.

RMRK: All subcollections of $\mathcal{B} X$.

Prop 19: $\mathcal{B} X$ is generated by any of the following : $E_1 = \{(a, b) : a < b\}$, $E_2 = \{[a, b] : a < b\}$, $E_3 = \{(a, b) : a < b\}$, $E_4 = \{(a, b) : a < b\}$, $E_5 = \{(a, \infty) : a \in \mathbb{R}\}$, $E_6 = \{(-\infty, a) : a \in \mathbb{R}\}$, or $E_7 = \{(-\infty, a) : a \in \mathbb{R}\}$. $\mathcal{B} X$ is the Borel $\sigma$-algebra on $\mathbb{R}$ can be generated by $\mathcal{E} = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E} = \{(-\infty, a) : a \in \mathbb{R}\}$. We also have $\mathcal{B} = \{E \subseteq \mathbb{R} : E \cap \mathbb{R} \in \mathcal{B} X\}$.

Prop 20: Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measure spaces. If $\mathcal{N} = \mathcal{M}(\mathcal{E})$ for some $\mathcal{E} \subseteq \mathcal{P}(X)$, then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Cor 21: If $f : X \to Y$, with $X$ and $Y$ topological spaces, is continuous, then $f$ is $(\mathcal{B} X, \mathcal{B} Y)$-measurable.

Prop 22: If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$ then, for all $a \in \mathbb{R}$, TFAE: (a) $f$ is measurable; (b) $f^{-1}((a, \infty)) \in \mathcal{M}$; (c) $f^{-1}([a, \infty)) \in \mathcal{M}$; (d) $f^{-1}((\infty, a)) \in \mathcal{M}$; (e) $f^{-1}((\infty, a)) \in \mathcal{M}$.

Def 23: 3 Let $\{(Y_\alpha, \mathcal{N}_\alpha)\}_{\alpha \in A}$ be a family of measurable spaces. If $f_\alpha : X_\alpha \to Y_\alpha$ is a map for each $\alpha \in A$, then the $\sigma$-algebra or $X$ generated by $\{f_\alpha\}_{\alpha \in A}$ is the $\sigma$-algebra generated by $\{f_\alpha^{-1}(E) : \alpha \in A \text{ and } E \in \mathcal{N}_\alpha\}$.

Prop 24: Let $(X, \mathcal{M})$ be a measurable space, and let $f, g : X \to \mathbb{R}$ be measurable. Then
(a) $|f|$ and $f^2$ are measurable;
(b) $\alpha f$, $\alpha + f$ are measurable for all $\alpha \in \mathbb{R}$;
(c) If $f$ is never 0, then $1/f$ is measurable;
(d) the set $\{x \in X : f(x) > g(x)\} \in \mathcal{M}$;
(e) $f + g$, $f - g$ are measurable;
(f) $f \cdot g$ is measurable;
(g) If $g$ is never 0 then $f/g$ is measurable.

Cor 25: If $f : X \to \mathbb{C}$ is measurable, then so is $\text{sgn}(f)$, where $\text{sgn}(z) = z/|z|$ if $z \neq 0$ and $0$ if $z = 0$.

Prop 26: If $(f_j)_{j=1}^\infty$ is a sequence of $\mathbb{R}$-valued functions on $(X, \mathcal{M})$ then $g_1(x) = \sup_{j \in \mathbb{N}} f_j(x)$, $g_2(x) = \inf_{j \in \mathbb{N}} f_j(x)$, $g_3(x) = \lim_{j \to \infty} f_j(x)$, and $g_4(x) = \liminf_{j \to \infty} f_j(x)$ are measurable. If $\lim_{j \to \infty} f_j(x) = f(x)$ then at each $x \in X$, $f(x)$ is measurable.

Cor 27: 4 If $f, g : X \to \mathbb{R}$ are measurable, then $x \mapsto \max\{f(x), g(x)\}$ and $x \mapsto \min\{f(x), g(x)\}$ are measurable.

Cor 28: If $f : X \to \mathbb{R}$ is measurable, then so are $f^+$, $f^-$, and $|f|$.

Def 29: A simple function on $X$ is a measurable function with finite range, which is a subset of $\mathbb{R}$ (or $\mathbb{C}$). In particular, a simple function is bounded. If $\varphi : X \to \mathbb{R}$ is simple, then $\text{Ran}(\varphi) = \{a_1, \cdots, a_n\}$, and for each $j \in \{1, \cdots, n\}$ the set $E_j = \varphi^{-1}(a_j)$ is measurable and $\bigcup_{j=1}^n E_j = X$. The standard representation for $\varphi$ is $\varphi = \sum_{j=1}^n a_j \chi_{E_j}$.
THM 30: Let \((X, \mathcal{M})\) be a measurable space. If \(f : X \to [0, \infty] \) is measurable, then there is a sequence \(\{\phi_j\}_{j=1}^{\infty}\) of simple functions such that (1) \(0 \leq \phi_1 \leq \phi_2 \leq \cdots\); (2) \(\lim_{j \to \infty} \phi_j(x) = f(x)\) for all \(x \in X\); and (3) \(\phi_j \to f\) uniformly on any set where \(f\) is uniformly bounded. (An analogous result holds if \(f\) has complex range.)

DEF 31: Let \((X, \mathcal{M})\) be a measurable space.

(a) A **(positive) measure** on the \(\sigma\)-algebra \(\mathcal{M}\) is a function \(\mu : \mathcal{M} \to [0, \infty]\) with \(\mu(\emptyset) = 0\) and if \(\{E_j\}_{j=1}^{\infty}\) is a sequence of mutually disjoint sets then \(\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)\).

(b) A **measure space** is a triple \((X, \mathcal{M}, \mu)\) with \(\mu\) a measure.

DEF 32: Let \((X, \mathcal{M}, \mu)\) be a measure space.

(a) \(\mu\) is called **semifinite** if for each \(E \in \mathcal{M}\) with \(\mu(E) > 0\) there is a subset \(F \in \mathcal{M}\) such that \(F \subseteq E\) and \(0 < \mu(F) < \infty\).

(b) \(\mu\) is called **\(\sigma\)-finite** if \(X = \bigcup_{j=1}^{\infty} E_j\) for some \(\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}\) and \(\mu(E_j) < \infty\) for all \(j \in \mathbb{N}\).

(c) A set \(E \in \mathcal{M}\) is called **\(\sigma\)-finite** if \(E = \bigcup_{j=1}^{\infty} E_j\) for some \(\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}\) with \(\mu(E_j) < \infty\) for all \(j \in \mathbb{N}\).

(d) \(\mu\) is called **finite** if \(\mu(X) < \infty\).

(e) \(\mu\) is called a **probability measure** if \(\mu(X) = 1\).

EXAMPLE 33:

(a) Suppose \(X\) is nonempty. Define \(\mu : \mathcal{P}(X) \to [0, \infty]\) by \(\mu(E) = \text{card}(E)\) if \(E\) is finite, and \(\mu(E) = \infty\) otherwise. Then \(\mu\) is semifinite, \(\mu\) is \(\sigma\)-finite iff \(\mu(E)\) is countable, and \(E \subseteq X\) is \(\sigma\)-finite iff \(E\) is countable.

(b) Let \(X\) be nonempty. Define \(\mathcal{M} = \{E \subseteq X : E\) or \(E^c\) is countable\} and \(\mu : \mathcal{M} \to [0, \infty]\) by \(\mu(E) = 0\) if \(E\) is countable and \(\mu(E) = 1\) if \(E\) is uncountable.

(c) Let \((X, \mathcal{M})\) be a measure space. Let \(\{x_0\} \in \mathcal{M}\) be given. Define \(\mu : \mathcal{M} \to [0, \infty]\) by \(\mu(E) = 1\) if \(x_0 \in E\) and \(\mu(E) = 0\) otherwise. This is called a **point mass** or **Dirac measure** at \(x_0\).

DEF 34: If \((X, \mathcal{M}, \mu)\) is a measure space, then a **\(\mu\)-null set**, or null set, in \(X\) is a set such \(E\) such that \(\mu(E) = 0\).

DEF 35: If statement \(P\) is true for all \(x \in X\) with \(\mu(X \setminus E) = 0\) then we say that \(P\) holds **almost everywhere** in \(X\).

DEF 36: Let \((X, \mathcal{M}, \mu)\) be a measure space. The measure \(\mu\) is called **complete** if whenever \(E \in \mathcal{M}\) is a null set we also find \(F \in \mathcal{M}\) for each \(F \subseteq E\).

THM 37: Suppose that \((X, \mathcal{M})\) is a measure space. Set \(\mathcal{N} := \{N \in \mathcal{M} : \mu(N) = 0\}\), and \(\mathcal{M} := \{E \cup F : E \in \mathcal{M}\) and \(F \subseteq N \in \mathcal{N}\}\). Then \(\mathcal{M}\) is a \(\sigma\)-algebra and there exists a unique extension of \(\mu\) to \(\mathcal{M}\), where \(\overline{\mu}\) is complete.

DEF 38: The measure space \((X, \overline{\mathcal{M}}, \overline{\mu})\) is called the **completion** of \((X, \mathcal{M}, \mu)\).

THM 39: Let \((X, \mathcal{M}, \mu)\) be a measure space. Then \(\mu\) has the following properties:

- (Monotonicity) If \(E, F \in \mathcal{M}\) and \(E \subseteq F\) then \(\mu(E) \leq \mu(F)\).
- (Countable subadditivity) If \(\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}\) then \(\mu(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)\).
- (Continuity from below) \(\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}\) and \(E_j \subseteq E_{j+1}\) for all \(j\) then \(\mu(\bigcup_{j=1}^{\infty} E_j) = \lim_{j \to \infty} \mu(E_j)\).
- (Continuity from above) If \(\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}\) and \(E_j \supseteq E_{j+1}\) and \(\mu(E_1) < \infty\) then \(\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j \to \infty} \mu(E_j)\).

2 Abstract Integration

**Notation:** Given a measurable space \((X, \mathcal{M})\), set \(L^+ := \{f : X \to [0, \infty] : f\) is measurable\}.

DEF 40: Let \((X, \mathcal{M}, \mu)\) be a measure space.
(a) Let φ ∈ L⁺ be a simple function with standard representation φ = ∑ₙ₌₁ⁿ⁻¹ aₖχₑₖ. The (Lebesgue) integral of φ with respect to μ is
\[ \int X φdμ := \sum_{j=1}^{n} a_jμ(E_j). \]

(b) Let f ∈ L⁺ be given. The (Lebesgue) integral of f with respect to μ is
\[ \int X fdμ := \sup \{ \int X φdμ : φ ∈ L⁺ and 0 ≤ φ ≤ f \}. \]

(c) Let f ∈ L⁺ and A ∈ ℳ. The (Lebesgue) integral of f with respect to μ over A is
\[ \int_A fdμ = \int_X fχ_A dμ. \]

PROP 41: Let f ∈ L⁺ be given.
(a) If A ∈ ℳ and μ(A) = 0 then \( \int_A f dμ = 0. \)
(b) If \( f(x) = 0 \) for a.e. \( x ∈ X \) then \( \int_X f dμ = 0. \)

PROP 42: Let φ ∈ L⁺ be a simple function. Define λ : ℳ → [0, +∞] by λ(E) := \( \int_E φdμ. \) Then λ is a measure.

PROP 43: Let φ, ψ ∈ L⁺ and c ∈ [0, ∞] be given, with φ, ψ simple. Then
(a) \( \int_X (cφ)dμ = c \int_X φdμ. \)
(b) \( \int_X (φ + ψ)dμ = \int_X φdμ + \int_X ψdμ. \)
(c) If \( φ(x) ≤ ψ(x) \) for a.e. \( x ∈ X \) then \( \int_X φdμ ≤ \int_X ψdμ. \)

COR 44: \(^8\) If \( f, g ∈ L⁺ \) and \( f ≤ g \) for a.e. \( x ∈ X \) then \( \int_X f dμ ≤ \int_X g dμ. \)

THM 45: (Monotone Convergence Theorem) Let \( \{f_j\}_{j=1}^{∞} ⊆ L⁺ \) satisfying \( 0 ≤ f_1 ≤ f_2 ≤ ··· ≤ f_j ≤ ···. \) Define \( f(x) = \lim_{j→∞} f_j(x) \) for all \( x ∈ X. \) Then \( f ∈ L⁺ \) and
\[ \int_X fdμ = \lim_{j→∞} \int_X f_j dμ. \]

COR 46: \(^9\) Suppose that \( \{a_{j,k}\}_{j,k=1}^{∞} ⊆ [0, +∞] \) satisfies \( 0 ≤ a_{j,1} ≤ a_{j,2} ≤ ··· \) for each \( j ∈ ℕ. \) Then
\[ \lim_{k→∞} \sum_{j=1}^{∞} a_{j,k} = \sum_{j=1}^{∞} \lim_{k→∞} a_{j,k}. \]

THM 47: If \( \{f_j\}_{j=1}^{∞} ⊆ L⁺ \) and \( f(x) = \sum_{j=1}^{∞} f_j(x) \) for each \( x ∈ X, \) then
\[ \int_X fdμ = \sum_{j=1}^{∞} \int_X f_j dμ. \]

PROP 48: If \( f ∈ L⁺ \) then \( \int_X fdμ = 0 \) if and only if \( f = 0 \) a.e.

COR 49: If \( \{f_j\}_{j=1}^{∞} ⊆ L⁺ \) and \( f ∈ L⁺ \) satisfy \( f_j(x) \) increases to \( f(x) \) for a.e. \( x ∈ X. \) then
\[ \int_X fdμ = \lim_{j→∞} \int_X f_j dμ. \]

LMA 50: (Fatou’s Lemma) If \( \{f_j\}_{j=1}^{∞} ⊆ L⁺ \) then
\[ \int_X (\liminf_{j→∞} f_j)dμ ≤ \liminf_{j→∞} \int_X f_jdμ. \]

Mike Janssen
Cor 51: If \( \{f_j\}_{j=1}^\infty \subseteq L^+ \) and \( \lim_{j\to\infty} f_j(x) = f(x) \) for a.e. \( x \in X \) then \( \int_X f \, d\mu \leq \liminf_{j\to\infty} \int_X f_j \, d\mu \).

Prop 52: If \( f \in L^+ \) and \( \int_X f \, d\mu < \infty \) then \( \{x \in X : f(x) = +\infty\} \) is a null set, and \( \{x \in X : f(x) > 0\} \) is \( \sigma \)-finite.

Def 53: Define \( L^1(X, \mathcal{M}, \mu) \) to be the collection of measurable functions \( f : X \to \mathbb{R} \) such that \( \int_X |f| \, d\mu < \infty \). Then we say that \( f \) is integrable, or \( \mu \)-integrable. If \( f \) is integrable, we define
\[
\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.
\]

Prop 54: If \( f \in L^1 \), then \( |\int_X f \, d\mu| \leq \int_X |f| \, d\mu \).

Prop 55: Let \( f, g \in L^1 \) be given. TFAE:

(a) \( \int_E f \, d\mu = \int_E g \, d\mu \) for all \( E \in \mathcal{M} \).
(b) \( \int_X |f - g| \, d\mu = 0 \)
(c) \( f = g \) \( \mu \)-a.e.

Def 56: We define \( L^1(X, \mu) \), or just \( L^1(X) \), \( L^1(\mu) \), or \( L^1 \), to be the collection of equivalence classes given in Proposition 55.

Prop 57: Suppose that \( \mu \) is a complete measure.

(a) If \( f \) is measurable and \( f = g \) \( \mu \)-a.e. then \( g \) is measurable.
(b) If \( \{f_j\}_{j=1}^\infty \) is a sequence of measurable functions and \( \lim_{j \to \infty} f_j = f \) exists \( \mu \)-a.e. \( x \in X \) then \( f \) is measurable.
(c) If \( f \) is \( \mu \)-measurable such that \( f = f \) \( \mu \)-a.e. then \( g \) is \( \mu \)-measurable.

(c) \( f = g \) \( \mu \)-a.e.

Def 58: \( (X, \mathcal{M}, \mathcal{L}, \mu) \) be the completion of \( (X, \mathcal{M}, \mu) \). If \( f : X \to \mathbb{R} \) is \( \mathcal{M} \)-measurable then there is \( g : X \to \mathbb{R} \) that is \( \mathcal{M} \)-measurable such that \( g = f \) \( \mathcal{L} \)-a.e. By convention, do not distinguish between \( L^1(\mu) \) and \( L^1(\mathcal{L}) \).

Def 59: Define \( \rho_1 : L^1(\mu) \times L^1(\mu) \to [0, +\infty] \) by \( \rho_1(f, g) = \int_X |f - g| \, d\mu \).

Prop 60: The function \( \rho_1 \) is a metric on \( L^1(\mu) \).

Thm 61: (Dominated Convergence Theorem) Let \( \{f_j\}_{j=1}^\infty \subseteq L^1(\mu) \) be a sequence satisfying

(a) \( \lim_{j \to \infty} f_j(x) = f(x) \) \( \mu \)-a.e. \( x \in X \).
(b) There is a \( g \in L^1(\mu) \) such that \( |f_j(x)| \leq g(x) \) for all \( j \in \mathbb{N} \) \( \mu \)-a.e. \( x \in X \).

Then \( f \in L^1(\mu) \) and
\[
\int_X f \, d\mu = \lim_{j \to \infty} \int_X f_j \, d\mu.
\]

Def 62: If \( \{f_j\}_{j=1}^\infty \subseteq L^1(\mu) \) and \( f \in L^1(\mu) \) satisfy \( \lim_{j \to \infty} \rho_1(f_j, f) = 0 \), then we write \( f_j \to f \) in \( L^1 \) and we say that \( f_j \) converge strongly to \( f \) in \( L^1 \).

Cor 63: Under hypotheses in DCT, we actually have that \( f_j \to f \) in \( L^1 \).

3 Some Functional Analysis

Def 64: A seminorm on \( X \) is a function \( \|\cdot\| : X \to [0, \infty) \) such that \( \|x + y\| \leq \|x\| + \|y\| \) for all \( x, y \in X \) and \( \|\lambda x\| = |\lambda| \cdot \|x\| \) for all \( \lambda \in E \) and \( x \in X \). If a seminorm satisfies \( \|x\| = 0 \) if and only if \( x = 0 \) then we say \( \|\cdot\| \) is a norm on \( X \).

Example 65:

(a) \( \mathbb{R}^n \) is a v.s. The function \( \|\cdot\|_p : \mathbb{R}^n \to [0, \infty) \) defined by \( \|x\|_p := \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \) for \( p \in [1, \infty) \) is a norm on \( \mathbb{R}^n \). Also, \( \|\cdot\|_\infty : \mathbb{R}^n \to [0, \infty) \) defined by \( \|x\|_\infty = \max\{|x_1|, \cdots, |x_n|\} \) is also a norm on \( \mathbb{R}^n \).

(b) Define \( \|\cdot\|_1 : L^1(\mu) \to [0, \infty) \) by \( \|f\|_1 := \int_X |f| \, d\mu \). It is easily verified that \( (L^1(\mu), \|\cdot\|_1) \) is a normed vector space.

Mike Janssen
Def 73: Two norms $||·||$ and $||·||_p$ on $X$ are **equivalent** if there are constants $C_1, C_2 > 0$ finite such that 
$C_1 ||x|| \leq ||x||_p \leq C_2 ||x||$ for all $x \in X$. Equivalent norms induce the same topologies.

Example 67: In $\mathbb{R}^n$, each of the norms $||·||_p$ for $p \in [1, \infty)$ are equivalent. In $\mathbb{R}^N$, the norms $||x||_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$ where $x = \{x_j\}_{j=1}^{\infty} \in \mathbb{R}^N$ are not equivalent as $p$ varies.

Def 68: If $(X, ||·||)$ is a complete normed vector space with respect to the metric $ρ||·||$ then we call $(X, ||·||)$, or just $X$, a **Banach space** (or a B-space).

Def 69: Let $(X, ||·||)$ be given. Let $\{x_j\}_{j=1}^{\infty} \subseteq X$ be given. The series $\sum_{j=1}^{\infty} x_j$ **converges** to $x \in X$ iff 
$\lim_{N \to \infty} \sum_{j=1}^{N} x_j = x$

with respect to the norm: i.e., 
$\lim_{N \to \infty} \left|\left| \sum_{j=1}^{N} x_j - x \right|\right| = 0$. We say that $\{x_j\}_{j=1}^{\infty}$ is absolutely convergent if 
$\sum_{j=1}^{\infty} ||x_j|| < \infty$.

Thm 70: (p. 152) A normed vector space is complete iff all absolutely convergent sequences converge in $X$.

Thm 71: The normed vector space $(L^1(\mu), ||·||)$ is a B-space.

Prop 72: The set of simple functions on $L^1(\mu)$ is dense in $L^1(\mu)$ with respect to the strong (norm) topology.

Def 73: For each $p \in (0, \infty)$, define $L^p(X, \mu)$ by $L^p(X, \mu) := \{f \text{ measurable : } \sup_{x \in X} |f(x)| < \infty\}$. Define $L^\infty(X, \mu) := \{f \text{ measurable : } \sup_{x \in X} |f(x)| < \infty\}$. Here, for $f : X \to \mathbb{R}$ measurable, then 
$\text{ess sup}_{x \in X} f(x) := \inf \{a \in \mathbb{R} : \mu(\{x \in X : f(x) > a\}) = 0\}$.

In other words, $f(x) \leq \text{ess sup}_{x \in X} f(x)$ a.e. and $\text{ess sup}_{x \in X} f(x)$ is the smallest number such that this inequality holds.

Def 74: For each $p \in (0, \infty)$, define $||·||_p : L^p(\mu) \to [0, \infty)$ by $||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p}$. Define $||·||_\infty : L^\infty(\mu, \mu) \to [0, \infty)$ by $||f||_\infty := \text{ess sup}_{x \in X} |f(x)|$. The triangle inequality is only satisfied if $p \geq 1$.

Thm 75: (Minkowski’s Inequality) Suppose that $p \in [1, \infty)$. Let $f, g \in L^+$ be given. Then

$\left(\int_X (f + g)^p d\mu\right)^{1/p} \leq \left(\int_X f^p d\mu\right)^{1/p} + \left(\int_X g^p d\mu\right)^{1/p}$.

Thm 76: (Hölder’s Inequality) Suppose that $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$. Let $f, g \in L^+$ be given.

$\int_X f g d\mu \leq \left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/q}$.

Thm 77: (Young’s Inequality) Suppose $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$. Let $a, b \in \mathbb{R}$ be given. Then $|ab| \leq (1/p)|a|^p + (1/q)|b|^q$.

Remark: It follows from Minkowski that the functions $||·||_p$ are norms in $L^p$ for $p \in [1, \infty)$. The case for $p = \infty$ can be proved directly.

Thm 78: The normed vector space $(L^p(\mu), ||·||_p)$ is a B-space for all $p \in [1, \infty]$.

Prop 79: For each $p \in [1, \infty]$, the simple functions are dense in $L^p$ with respect to the norm topology.

Prop 80: If $1 \leq p \leq q \leq r \leq \infty$ then $L^p \cap L^r \subseteq L^q$ and $||f||_q \leq ||f||_p ||f||_r^{1-\lambda}$ where $1/q = \lambda/p + (1 - \lambda)/r$.

Prop 81: If $\mu$ is finite and $1 \leq p \leq q \leq \infty$, then $L^q \subseteq L^p$ and $||f||_p \leq ||f||_q \mu(X)^{1/p-1/q}$.

Modes of Convergence

- **Uniformly** in $X$ if for all $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$ we have $|f_n(x) - f(x)| < \varepsilon$.
- **Pointwise** if for each $x \in X$ and $\varepsilon > 0$ there is $N(\varepsilon, x) \in \mathbb{N}$ such that $|f_j(x) - f(x)| < \varepsilon$.
- **Convergence in $L^p$ for $p \in [1, \infty)$** We say $f_j \to f$ in $L^p(X, \mu)$ if for each $\varepsilon > 0$ there is $N(\varepsilon) \in \mathbb{N}$ such that $||f_j - f||_p < \varepsilon$ for all $j \geq N(\varepsilon)$.
- **Almost Everywhere** in $X$ if there is some $\mu$-null set $E$ such that $f_j \to f$ pointwise in $X \setminus E$.
- **Convergence in Measure** if for each $\varepsilon > 0$ we have $\lim_{j \to \infty} \mu(\{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}) = 0$.

Mike Janssen
• **Cauchy in measure**\(^{18}\): We say that that \(\{f_j\}_{j=1}^{\infty}\) is **Cauchy in measure** if for each \(\varepsilon > 0\), 
\[
\lim_{j,k \to \infty} \mu(\{x \in X : |f_j(x) - f_k(x)| \geq \varepsilon\}) = 0.
\]

**Thm 82**: (Egoroff’s Theorem) Suppose that \(\mu\) is finite and \(\{f_j\}_{j=1}^{\infty}\) and \(f\) are measurable functions such that \(f_j \to f\) a.e. in \(X\). Then for each \(\alpha > 0\) there is a measurable set \(E \subseteq X\) such that \(\mu(E) < \alpha\) and \(f_j \to f\) uniformly on \(X \setminus E\).

**Thm 83**: Suppose that \(\{f_j\}_{j=1}^{\infty}\) is a sequence of measurable functions that are Cauchy in measure. Then there exists a measurable \(f\) such that \(f_j \to f\) in measure.

**Thm 84**:\(^{19}\) If \(\{f_j\}_{j=1}^{\infty}\) is a sequence of measurable functions such that \(f_j \to f\) in measure, with \(f\) measurable, then there is a subsequence \(\{f_{j_k}\}_{k=1}^{\infty} \subseteq \{f_j\}_{j=1}^{\infty}\) such that \(f_{j_k} \to f\) for \(\mu\)-a.e. \(x \in X\).

**Thm 85**: If \(\{f_j\}_{j=1}^{\infty}\) is a sequence of measurable functions such that \(f_j \to f\) in measure and \(f_j \to g\) in measure for measurable \(f, g\), then \(f = g\) for \(\mu\)-a.e. \(x \in X\).

**Thm 86**: Let \(p \in [1, \infty]\) be given. If \(\{f_j\}_{j=1}^{\infty} \subseteq L^p(\mu)\) such that \(f_j \to f\) in \(L^p\) for some \(f \in L^p\) then there is a subsequence \(\{f_{j_k}\}_{k=1}^{\infty} \subseteq \{f_j\}_{j=1}^{\infty}\) such that \(f_{j_k} \to f\) for \(\mu\)-a.e. \(x \in X\).

**Example 87**: Let \(X = \mathbb{N}\), \(\mathcal{M} = \mathcal{P}(\mathbb{N})\), and \(\mu\) the counting measure. So for \(f : \mathbb{N} \to \mathbb{R}\) we have \(\int_X f \, d\mu = \sum_{k=1}^{\infty} f(k)\) (assuming the series converges absolutely). Suppose that \(f_j(k) = k/j\) for all \(k, j \in \mathbb{N}\). Then \(f_j(k) \to 0\) pointwise in \(\mathbb{N}\), \(f_j \not\to 0\) uniformly, given \(p \in [1, \infty)\), \(\int_{\mathbb{N}} |f_j - 0|^p \, d\mu = 1/j^p \sum_{k=1}^{\infty} k^p\) diverges, and \(f_j \not\to 0\) in measure.

## 4 Construction of Measures

### 4.1 Carathéodory’s Approach

**Thm (Ulam)**\(^{20}\) Suppose that \(\mu\) is a measure on \(\mathbb{R}\) such that every subset \(E \subseteq \mathbb{R}\) is measurable. If \(\mu([n, n+1)) < \infty\) for each \(n \in \mathbb{Z}\) and \(\mu([x]) = 0\) for each \(x \in \mathbb{R}\), then \(\mu(E) = 0\) for all \(E \subseteq \mathbb{R}\).

**Def 88**: A family of sets \(E \subseteq \mathcal{P}(X)\) is a **semi-algebra** (or **elementary family**) on \(X\) if it has the following properties:

1. \(\emptyset, X \in E\)
2. If \(E_1, E_2 \in E\) then \(E_1 \cap E_2 \in E\)
3. If \(E \in E\) then \(E^c = \bigcup_{i=1}^{\infty} E_i\) where \(E_i\) is a finite disjoint sequence of sets in \(E\).

**Example 89**:

1. \(\mathcal{I} := \\{\text{open, half-closed, half-open, closed intervals in } \mathbb{R}\}\).
2. \(\mathcal{T}^n := \{\text{cartesian product of } n \text{ elements of } \mathcal{I}\}\).

**Notation**: \(I(a, b)\) will be used to denote any one of the intervals \((a, b), [a, b], [a, b), \text{ or } (a, b]\), for all \(a, b \in \mathbb{R}\) which make sense. Define \(\mathcal{I} = \{I(a, b) : a \leq b, a, b \in \mathbb{R}\}\) and \(\mathcal{T}^n = \left\{\prod_{j=1}^{n} I(a_j, b_j) : a_j \leq b_j \text{ for } j = 1, \ldots, n\right\}\).

**Def 90**:\(^{21}\) Given \(n \in \mathbb{N}\), define \(m^n : \mathcal{T}^n \to [0, \infty]\) by \(m^n \left(\prod_{j=1}^{n} I(a_j, b_j)\right) = \prod_{j=1}^{n} (b_j - a_j)\). When \(n\) is understood, we write \(m\) for \(m^n\).

**Def 91**: Let \(E \subseteq \mathcal{P}(X)\) be a semi-algebra. A set function \(\mu : E \to [0, \infty]\) is called

- **monotone** if \(\mu(E) \leq \mu(F)\) whenever \(E, F \in E\) and \(E \subseteq F\).
- **finitely additive** if \(\mu(\bigcup_{j=1}^{n} E_j) = \sum_{j=1}^{n} \mu(E_j)\) whenever \(\{E_j\}_{j=1}^{n} \subseteq E\) is a sequence of mutually disjoint sets with \(\bigcup_{j=1}^{n} E_j \in E\).
- **countably additive** if \(\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)\) whenever \(\{E_j\}_{j=1}^{\infty} \subseteq E\) are mutually disjoint and \(\bigcup_{j=1}^{\infty} E_j \in E\).
- **countably subadditive** if \(\mu(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)\) whenever \(\{E_j\}_{j=1}^{\infty} \subseteq E\) and \(\bigcup_{j=1}^{\infty} E_j \in E\).

Mike Janssen
Example 92: It is possible to show that $m^n$ is monotone, finitely additive, and countably subadditive (and countably additive, but this is hard to show).

Def 93: A family of sets $A \subseteq \mathcal{P}(X)$ is called an algebra if

(a) $\emptyset, X \in A$
(b) If $E_1, E_2 \in A$ then $E_1 \cap E_2 \in A$
(c) If $E \in A$ then $E^c \in A$.

Prop 94: Let $C \subseteq \mathcal{P}(X)$ be given. Then there exists a unique smallest algebra $\mathcal{A}(C) \subseteq \mathcal{P}(X)$ such that $C \subseteq \mathcal{A}(C)$ and if $F \subseteq \mathcal{P}(X)$ is an algebra containing $C$ then $\mathcal{A}(C) \subseteq F$.

Def 95: The algebra provided by Proposition 94 is called the algebra generated by $C$.

Prop 96: Suppose that $\mathcal{E} \subseteq \mathcal{P}(X)$ is a semi-algebra. Then

$$\mathcal{A}(\mathcal{E}) = \left\{ E \subseteq X : E = \bigcup_{j=1}^{n} E_j \text{ for some disjoint sequence } \{E_j\}_{j=1}^{n} \subseteq \mathcal{E} \right\}$$

Example 97: $\mathcal{A}(I) = \{ \text{finite unions of intervals in } \mathbb{R} \}$, and $\mathcal{A}(I^n) = \{ \text{finite unions of boxes in } \mathbb{R}^n \}$.

Thm 98: Suppose that $\mu$ is a finitely-additive, countably subadditive set function on a semi-algebra $\mathcal{E}$ such that $\mu(\emptyset) = 0$.

Then there exists a unique countably subadditive set function $\hat{\mu}$ on $\mathcal{A}(\mathcal{E})$ extending $\mu$.

Thm 99: Let $A$ be an algebra of sets and let $\hat{\mu} : A \rightarrow [0, \infty]$ such that $\hat{\mu}(\emptyset) = 0$ be given. Then $\hat{\mu}$ is countably additive if and only if $\hat{\mu}$ is finitely additive and countably subadditive.

Def 100: Suppose that $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra. A function $\hat{\mu} : A \rightarrow [0, \infty]$ is a premeasure on $\mathcal{A}$ if

(a) $\hat{\mu}(\emptyset) = 0$
(b) $\hat{\mu}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} (E_j)$ whenever $\{E_j\}_{j=1}^{\infty}$ is mutually disjoint and $\cup_{j=1}^{\infty} E_j \subseteq A$.

Notation: Denote the left-open and right-closed intervals in $\mathbb{R}$ by $\mathcal{H}$, so

$$\mathcal{H} := \{(a,b) : (a,b] \subseteq \mathbb{R}\} \cup \{(a,\infty) : a \in \mathbb{R}\}.$$  

Now, $\mathcal{H}$ is a semi-algebra, and $\mathcal{B}_\mathbb{R}$ will be generated by $\mathcal{H}$. Given a function $F : \mathbb{R} \rightarrow \mathbb{R}$, define $F(\infty) := \lim_{x \rightarrow \infty} F(x)$ and $F(-\infty) := \lim_{x \rightarrow -\infty} F(x)$, provided the limits exist in $\mathbb{R}$.

Prop 101: Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Define $\mu_F : \mathcal{H} \rightarrow [0, \infty]$ by $\mu_F((a,b]) := F(b) - F(a)$ if $(a,b] \subseteq \mathbb{R}$ and $\mu_F((a,\infty)) := F(\infty) - F(a)$ if $a \in [\infty, \infty)$. Then $\mu_F$ is well-defined. Moreover, if $\mu_F$ is right-continuous, then $\mu_F$ is countably subadditive.

Prop 102: Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing and right-continuous. Define $\mu_F : \mathcal{H} \rightarrow [0, \infty]$ as in Proposition 101.

Then $\hat{\mu}_F : \mathcal{A}(\mathcal{H}) \rightarrow [0, \infty]$ defined by $\hat{\mu}_F(\bigcup_{j=1}^{n} I_j) = \sum_{j=1}^{n} \mu_F(I_j)$ whenever $\{I_j\}_{j=1}^{n} \subseteq \mathcal{H}$ is mutually disjoint is a premeasure on $\mathcal{A}(\mathcal{H})$.

Prop 103: Suppose that $\mu : \mathcal{H} \rightarrow [0, \infty]$ is finitely additive and $\mu((a,b]) < \infty$ for each $a, b \in \mathbb{R}$. Then there is a function $F : \mathbb{R} \rightarrow \mathbb{R}$ that is nondecreasing such that $\mu((a,b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$. If $\mu$ is countably additive, then $F$ is right-continuous and $\mu = \mu_F$.

Def 104: An outer measure on $X$ is a monotone countably subadditive set function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ such that $\mu^*(\emptyset) = 0$.

Prop 105: Let $\mathcal{E} \subseteq \mathcal{P}(X)$ and $\rho : \mathcal{E} \rightarrow [0, \infty]$ such that $\emptyset, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For each $E \subseteq X$ define

$$\mu^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : \{E_j\} \subseteq \mathcal{E}, E \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$  

Then $\mu^*$ is an outer measure.

Def 106: We call $\mu^*$ in Proposition 105 the outer measure induced by $\rho$.

Def 107: If $\mu^*$ is an outer measure on $X$ then a set $A \subseteq X$ is called $\mu^*$-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

Mike Janssen
4.1 Carathéodory’s Approach

**Thm 108:** (Carathéodory’s Theorem) If \( \mu^* \) is an outer measure on \( X \), then the collection \( \mathcal{M} \) of \( \mu^* \)-measurable sets is a \( \sigma \)-algebra. Moreover, the restriction of \( \mu^* \) to \( \mathcal{M} \) is a complete measure.

**Prop 109:** If \( \tilde{\mu} \) is a premeasure on an algebra \( \mathcal{A} \subseteq \mathcal{P}(X) \) and \( \mu^* \) is the outer measure induced by \( \tilde{\mu} \) then \( \mu^*|_\mathcal{A} = \tilde{\mu} \) and every set of \( \mathcal{A} \) is \( \mu^* \)-measurable.

**Thm 110:** Let \( \mathcal{A} \subseteq \mathcal{P}(X) \) be an algebra and let \( \mathcal{M}(\mathcal{A}) \) be the \( \sigma \)-algebra generated by \( \mathcal{A} \). Let \( \tilde{\mu} \) be a premeasure on \( \mathcal{A} \).

(a) There is a measure \( \mu \) on \( \mathcal{M}(\mathcal{A}) \) such that \( \mu|_\mathcal{A} = \tilde{\mu} \). (In fact, \( \mu = \mu^*|_{\mathcal{M}(\mathcal{A})} \).

(b) If \( \nu \) is another measure extending \( \tilde{\mu} \) to \( \mathcal{M}(\mathcal{A}) \), then \( \nu(E) \leq \mu(E) \) for all \( E \in \mathcal{M}(\mathcal{A}) \), and \( \mu(E) = \nu(E) \) whenever \( \mu(E) < \infty \).

(c) If \( \tilde{\mu} \) is \( \sigma \)-finite then \( \mu \) is the unique extension of \( \tilde{\mu} \) to a measure on \( \mathcal{M}(\mathcal{A}) \).

**Rmrk:** Recall (Proposition 102) that if \( F : \mathbb{R} \to \mathbb{R} \) is a non-decreasing right-continuous function then \( \tilde{\mu}_F : \mathcal{A}(\mathcal{H}) \to [0, \infty] \) is a premeasure on \( \mathcal{A}(\mathcal{H}) \). Since \( \tilde{\mu}_F \) is \( \sigma \)-finite on \( \mathbb{R} \), there is a unique extension of \( \tilde{\mu}_F \) to a measure \( \mu_F \) on the \( \sigma \)-algebra generated by \( \mathcal{A}(\mathcal{H}) \), which is \( \mathcal{B}_\mathbb{R} \). Actually, Carathéodory’s Theorem gives a complete measure on the \( \sigma \)-algebra of \( \mu_F^* \)-measurable sets. Usually we use \( \mu_F \) for both the measure on \( \mathcal{B}_\mathbb{R} \) and the measure on \( \mathcal{M}(\mathcal{B}_\mathbb{R}) \) (the \( \mu_F^* \)-measurable sets). The complete measure \( \mu_F \) on \( \mathcal{M}(\mathcal{B}_\mathbb{R}) \) is called the Lebesgue-Stieltjes measure associated with \( F \).

If \( E \in \mathcal{M}(\mathcal{B}_\mathbb{R}) \), then

\[
\mu_F(E) = \inf \left\{ \sum_{j=1}^{\infty} |F(b_j) - F(a_j)| : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.
\]

In fact,

\[
\mu_F(E) = \inf \left\{ \sum_{j=1}^{\infty} |F(b_j) - F(a_j)| : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.
\]

**Def 111:** Let \( \mu_F \) be a Lebesgue-Stieltjes measure on \( \mathcal{M}(\mathcal{B}_\mathbb{R}) \). If \( g \in L^1(\mu_F) \) then \( \int g \, d\mu_F \) is called the Lebesgue-Stieltjes integral of \( g \) with respect to \( \mu_F \).

**Thm 112:** If \( F : \mathbb{R} \to \mathbb{R} \) is non-decreasing and continuously differentiable, and \( g : \mathbb{R} \to \mathbb{R} \) is continuous, then

\[
\int_{(a,b]} g \, d\mu_F = \int_a^b gF' \, dx.
\]

**Cor 113:** If \( F(x) = x \), then \( \mu_F = m \) and

\[
\int_{(a,b]} g \, dm = \int_a^b g \, dx.
\]

**Def 114:** Let \( \mu \) be a Borel measure on \( X \) and let \( E \in \mathcal{B}_X \) be given. The measure \( \mu \) is **outer regular** on \( E \) if

\[
\mu(E) = \inf \{ \mu(U) : E \subseteq U \text{ and } U \text{ is open} \}.
\]

The measure \( \mu \) is **inner regular** on \( E \) if

\[
\mu(E) = \sup \{ \mu(K) : K \subseteq E \text{ and } K \text{ is compact} \}.
\]

\( \mu \) is called **regular** if it is both inner regular and outer regular on all \( E \in \mathcal{B}_X \).

**Def 115:** A Borel measure \( \mu \) on \( X \) is called a **Radon measure** if \( \mu \) is

(a) finite on all compact sets

(b) outer regular on all \( E \in \mathcal{B}_X \)

(c) inner regular on all open sets.
4 CONSTRUCTION OF MEASURES

**Hausdorff Measures**

We seek a way to measure low-dimensional subsets of \( \mathbb{R}^n \). Note that the Lebesgue measure of a lower-dimensional subset of \( \mathbb{R}^n \) is always 0. There is a continuum of Hausdorff measures in \( \mathbb{R}^n \) that can be used to provide a refined measure of Lebesgue null sets in \( \mathbb{R}^n \).

Let \((X, \rho)\) be a metric space.

**Def 121:** Let \( H_p, H_p : \mathcal{P}(X) \to [0, \infty] \) by

\[
H_{p, \delta}(E) := \inf \left\{ \sum_{j=1}^{\infty} (\text{diam}(E_j))^p : E \subseteq \bigcup_{j=1}^{\infty} E_j, \text{diam}(E_j) \leq \delta \right\}
\]

and

\[
H_p(E) := \lim_{\delta \to 0^+} H_{p, \delta}(E).
\]

The function \( H_p \) is called the \( p \)-dimensional Hausdorff (outer) measure.

**Rmrk:** In the definition of \( H_{p, \delta} \), the sets \( E_j \) can be required to be open or closed.

**Rmrk:** \( H_0 \) is the counting measure.

**Rmrk:** In \( \mathbb{R}^n \), the \( n \)-dimensional volume of an \( m \)-dimensional ball \( b \) is proportional to \( (\text{diam}(B))^m \). In fact, the volume of an \( m \)-dimensional ball with the same diameter as an \( n \)-dimensional ball \( C \) is also proportional to \( (\text{diam}(C))^m \).

**Def 122:** An outer measure \( \mu^* \) on \((X, \rho)\) is called a metric outer measure if \( \mu^*(A \cup B) = \mu^*(A) + \mu^*(B) \) whenever \( \rho(A, B) > 0 \). This is clearly more than disjointness. Recall \( \rho(A, B) = \inf \{ \rho(x, y) : x \in A, y \in B \} \).

**Prop 123:** If \( \mu^* \) is a metric outer measure, then every Borel set is \( \mu^* \)-measurable.

**Prop 124:** Let \( A \) be a Borel set. If \( H_p(A) < \infty \), then \( H_q(A) = 0 \) for all \( q > p \). If \( H_p(A) > 0 \) then \( H_q(A) = \infty \) for all \( q < p \). It follows that \( \inf \{ p \geq 0 : H_p(A) = 0 \} = \sup \{ p \geq 0 : H_p(A) = \infty \} \). This value is called the Hausdorff dimension of \( A \).

Mike Janssen
4.3 Product Spaces

**Def 125:** Let \( \{X_\alpha, \mathcal{M}_\alpha\}_{\alpha \in A} \) be measurable spaces. Then the product \( \sigma \)-algebra on \( X = \prod_{\alpha \in A} X_\alpha \) is the \( \sigma \)-algebra generated by \( \{ \pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A \} \) where \( \pi_\alpha : X \to X_\alpha \) is the \( \alpha \)-th coordinate map.

**Notation:** This product \( \sigma \)-algebra is denoted by \( \bigotimes_{\alpha \in A} \mathcal{M}_\alpha \).

**Prop 126:** If \( A \) is countable, then \( \bigotimes_{\alpha \in A} \mathcal{M}_\alpha \) is the \( \sigma \)-algebra generated by \( \left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha \right\} \).

**Prop 127:** Let \( X_1, \cdots, X_n \) be metric spaces. Let \( X = \prod_{j=1}^n X_j \) equipped with the product metric. Then \( \bigotimes_{j=1}^n \mathcal{B}_{X_j} \subseteq \mathcal{B}_X \).

If each \( X_j \) is separable, then \( \bigotimes_{j=1}^n \mathcal{B}_{X_j} = \mathcal{B}_X \).

**Cor 128:** \( \mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^n \mathcal{B}_{\mathbb{R}} \).

Let \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) be measure spaces.

**Def 129:** If \( E \subseteq \mathcal{M}, F \subseteq \mathcal{N} \), then \( E \times F \) is a measurable rectangle. We denote the set of measurable rectangles by \( \mathcal{R} \). Note that \( \mathcal{M} \otimes \mathcal{N} \) is the \( \sigma \)-algebra generated by \( \mathcal{R} \) by Proposition 126. Observe that \( \mathcal{R} \) is a semi-algebra.

**Thm 130:** Let \( \pi : \mathcal{R} \to [0, \infty] \) be defined by \( \pi(E \times F) = \mu(E)\nu(F) \). Then \( \pi \) is well-defined, countably additive, and \( \pi(\emptyset) = 0 \).

**Thm 131:** There exists a unique extension of \( \pi \) to a premeasure \( \tilde{\pi} \) on \( \mathcal{A}(\mathcal{R}) \).

**Thm 132:** The premeasure \( \tilde{\pi} \) generates an outer measure \( \pi^* \) on \( X \times Y \). The restriction of \( \pi^* \) to \( \mathcal{M} \otimes \mathcal{N} \) is a measure extending \( \pi \). Moreover, if \( \mu \) and \( \nu \) are \( \sigma \)-finite then so is the measure \( \pi^*|_{\mathcal{M} \otimes \mathcal{N}} \). This is the unique extension of \( \tilde{\pi} \) to a measure on \( \mathcal{M} \otimes \mathcal{N} \).

**Notation:** We denote the measure \( \pi^*|_{\mathcal{M} \otimes \mathcal{N}} \) by \( \mu \times \nu \).

**Question:** How does one measure \( E \in \mathcal{M} \otimes \mathcal{N} \)?

**Def 133:** If \( E \subseteq X \times Y \), then for each \( x \in X \) then the \( x \)-section of \( E \) is \( E_x := \{ y \in Y : (x, y) \in E \} \). For each \( y \in Y \), the \( y \)-section of \( E \) is \( E^y := \{ x \in X : (x, y) \in E \} \). If \( f : X \times Y \to \mathbb{R} \) then the \( x \)-section \( f_x \) and \( y \)-section \( f^y \) of \( f \) are \( f_x(y) = f(x, y) \) and \( f^y(x) = f(x, y) \).

**Prop 134:**

(a) If \( E \in \mathcal{M} \otimes \mathcal{N} \), then \( E_x \in \mathcal{N} \) for all \( x \in X \) and \( E^y \in \mathcal{M} \) for all \( y \in Y \).

(b) If \( f \) is \( \mathcal{M} \otimes \mathcal{N} \)-measurable then \( f_x \) is \( \mathcal{N} \)-measurable and \( f^y \) is \( \mathcal{M} \)-measurable for each \( x \in X, y \in Y \), respectively.

**Def 135:** A collection \( \mathcal{C} \subseteq \mathcal{P}(X) \) is a monotone class if it has the following properties:

(a) If \( \{E_j\}_{j=1}^\infty \subseteq \mathcal{C} \) and \( E_1 \subseteq E_2 \subseteq \cdots \), then \( \cup_{j=1}^\infty E_j \in \mathcal{C} \).

(b) If \( \{E_j\}_{j=1}^\infty \subseteq \mathcal{C} \) and \( E_1 \supseteq E_2 \supseteq \cdots \), then \( \cap_{j=1}^\infty E_j \in \mathcal{C} \).

**Note:** All \( \sigma \)-algebras are monotone classes. Given any \( \mathcal{E} \subseteq \mathcal{P}(X) \), there is a unique smallest monotone class, \( \mathcal{C}(\mathcal{E}) \), called the monotone class generated by \( \mathcal{E} \).

**Thm 136:** \( \mathcal{M} \subseteq \mathcal{P}(X) \) is a \( \sigma \)-algebra if and only if \( \mathcal{M} \) is a monotone class and an algebra.

**Fact:** If \( E \in \mathcal{M} \otimes \mathcal{N} \), we want

\[
\mu \times \nu(E) = \int_X \nu(E_x) \, d\mu(x) = \int_Y \mu(E^y) \, d\nu(y).
\]

**Lma 137:** (Monotone Class Lemma) If \( \mathfrak{A} \subseteq \mathcal{P}(X) \) is an algebra, then the monotone class \( \mathcal{C}(\mathfrak{A}) \) and the \( \sigma \)-algebra \( \mathcal{M}(\mathfrak{A}) \) are equal.

**Thm 138:** Suppose that \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) are \( \sigma \)-finite spaces. Let \( E \in \mathcal{M} \otimes \mathcal{N} \) be given. Then:

(i) \( x \mapsto \nu(E_x) \) and \( y \mapsto \mu(E^y) \) are measurable, and
(ii) We have
\[ \mu \times \nu(E) = \int_X \nu(E_x) \, d\mu(x) \]
\[ = \int_Y \mu(E^y) \, d\nu(y). \]

**Thm 139: (Fubini-Tonelli)** Suppose \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) are \(\sigma\)-finite.

(a) (Tonelli) If \(f \in L^+(X \times Y)\) then \(g(x) = \int_Y f_x \, d\nu\) and \(h(y) = \int_X f^y \, d\mu\) are in \(L^+(X)\) and \(L^+(Y)\), respectively, and
\[
\int_{X \times Y} f(x, y) \, d(\mu \times \nu) = \int_X \int_Y f \, d\nu \, d\mu = \int_Y \int_X f \, d\mu \, d\nu.
\]

(b) (Fubini) If \(f \in L^1(\mu \times \nu)\) then \(f_x \in L^1(\nu)\) and \(f^y \in L^1(\mu)\) for a.e. \(x \in X\) and a.e. \(y \in Y\). Moreover, the functions \(g\) and \(h\) from (a) are in \(L^1(\mu)\) and \(L^1(\nu)\), respectively, and (1) holds.

**Rmrk:** To use (b), you need to show \(f \in L^1\). The common method for doing so is as follows: If \(f\) is measurable, then \(|f| \in L^+\), and so (a) implies
\[
\int_{X \times Y} |f| \, d(\mu \times \nu) = \int_X \int_Y |f| \, d\nu \, d\mu = \int_Y \int_X |f| \, d\mu \, d\nu,
\]
and then show that one of the last two expressions is finite.

### 5 Decomposition of Measures

#### 5.1 Intro and Definitions

Let \((X, \mathcal{M})\) be a measurable space.

**Def 140:** Let \(\mu, \nu\) be measures on \(\mathcal{M}\).

(a) \(\mu\) and \(\nu\) are **mutually singular**, and we write \(\mu \perp \nu\), if there are disjoint sets \(X_\mu\) and \(X_\nu\) in \(\mathcal{M}\) such that \(X = X_\mu \cup X_\nu\) and \(\mu(E) = \mu(E \cap X_\mu)\) and \(\nu(E) = \nu(E \cap X_\nu)\) for all \(E \in \mathcal{M}\).

(b) We say that \(\nu\) is **absolutely continuous with respect to** \(\mu\), and we write \(\nu \ll \mu\), if for every \(E \in \mathcal{M}\), we have \(\nu(E) = 0\) whenever \(\mu(E) = 0\).

(c) \(\nu\) is **diffuse with respect to** \(\mu\) if for each \(E \in \mathcal{M}\) we have \(\nu(E) = 0\) whenever \(\mu(E) < \infty\).

**Rmrk:**

(a) If \(\mu \perp \nu\), then \(\mu(X_\nu) = 0\) and \(\nu(X_\mu) = 0\).

(b) If \(\nu\) is diffuse with respect to \(\mu\), then \(\nu \ll \mu\).

(c) If \(f \in L^+\), then \(\nu : \mathcal{M} \to [0, \infty]\) defined by
\[
\nu(E) := \int_E f \, d\mu
\]
(2)
is a measure. By Proposition 41, we have that \(\nu \ll \mu\). It turns out that (2) characterizes all measures that are absolutely continuous with respect to \(\mu\) if \(\mu\) is \(\sigma\)-finite, i.e., \(\nu \ll \mu\) iff (2) holds for some \(f \in L^+\).
Thm 141: Let $\mu, \nu$ be measures on $\mathcal{M}$ with $\nu$ finite. Then $\nu \ll \mu$ iff for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\nu(E) < \varepsilon$ whenever $E \in \mathcal{M}$ and $\mu(E) < \delta$.

Cor 142: If $f \in L^1(\mu)$, then for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $\mu(E) < \delta$ then
\[
\left| \int_E f \, d\mu \right| < \varepsilon.
\]

**Notation:** If $\nu(E) = \int_E f \, d\mu$, with $f \in L^+(\mu)$, then we may write $\frac{d\nu}{d\mu}$ for $f$, or we may write $d\nu$ for $f \, d\mu$. Sometimes we will refer to $d\nu$ as a measure.

Thm 143: (Radon-Nikodym Theorem I) Let $\mu$ and $\nu$ be measures on $\mathcal{M}$. Suppose that $\mu$ is $\sigma$-finite and $\nu \ll \mu$. Then there is an $f \in L^+$ such that $\nu(E) = \int_E f \, d\mu$ for each $E \in \mathcal{M}$. The function $f$ is unique up to a $\mu$-null set.

Lma 144: Let $\mu, \nu$ be measures on $\mathcal{M}$. Define $\nu_{ac} : \mathcal{M} \to [0, +\infty]$ by
\[
\nu_{ac}(E) := \sup \left\{ \int_E f \, d\mu : f \in L^+ \text{ and } \int_E f \, d\mu \leq \nu(E) \forall F \subseteq E, F \in \mathcal{M} \right\}.
\]
Then $\nu_{ac}$ is a measure, $\nu_{ac} \ll \mu$, and, for each $E \in \mathcal{M}$ there is an admissible $f$ such that $\nu_{ac}(E) = \int_E f \, d\mu$. Moreover, if $\nu_{ac}$ is $\sigma$-finite, then $f$ can be chosen independently of $E$.

Lma 145: Let $\mu, \nu$ be finite measures on $\mathcal{M}$. For each $E \in \mathcal{M}$, define
\[
(\nu - \mu)^+(E) := \sup \{ \nu(F) - \mu(F) : F \subseteq E \text{ and } F \in \mathcal{M} \}.
\]
Then $(\nu - \mu)^+$ is a measure, and for each $E \in \mathcal{M}$ we have that
\[
(\nu - \mu)^+(E) := \sup \{ \nu(F) - \mu(F) : F \subseteq E, F \in \mathcal{M}, \text{ and } (\nu - \mu)^+(F) = 0 \}.
\]

Def 146: Let $\mathcal{F} = \{ f_\alpha \}_{\alpha \in A}$ be a family of measurable functions $f_\alpha : X \to \mathbb{R}$. We call $f : X \to \mathbb{R}$ an essential supremum of $\mathcal{F}$ if
(a) $f$ is measurable
(b) $f(x) \geq f_\alpha(x)$ for each $\alpha \in A$ and a.e. $x \in X$
(c) $f(x) \leq g(x)$ a.e. if $g$ satisfies a and b.

Def 147: Let $\{ E_\alpha \}_{\alpha \in A} \subseteq \mathcal{M}$ be given. A set $E \in \mathcal{M}$ is called an essential union of $\{ E_\alpha \}_{\alpha \in A}$ if $\chi_E$ is an essential supremum of $\{ \chi_{E_\alpha} \}_{\alpha \in A}$.

Rmrk: It can be shown that if $\{ \chi_{E_\alpha} \}_{\alpha \in A}$ has an essential supremum then $\{ E_\alpha \}_{\alpha \in A}$ has an essential union.

Def 148: A measure $\mu$ on $\mathcal{M}$ is localizable if any family of measurable sets has an essential union.

Rmrk: Any $\sigma$-finite measure is localizable.

Thm 149: Let $\mu$ be a $\sigma$-finite measure on $\mathcal{M}$. Let $\mathcal{F} = \{ f_\alpha \}_{\alpha \in A}$ be a family of measurable functions, $f_\alpha : X \to \mathbb{R}$. Then there is a countable set $I \subseteq A$ such that the function $x \mapsto \sup_{\alpha \in I} f_\alpha(x)$ is an essential supremum.

Thm 150: (Radon-Nikodym II) Let $\mu, \nu$ be measures on $\mathcal{M}$. Suppose that $\mu$ is localizable, $\nu \ll \mu$ and that
\[
\nu(E) = \sup \{ \nu(E \cap F) : F \in \mathcal{M} \text{ and } \mu(F) < \infty \}
\]
for all $E \in \mathcal{M}$. Then for all $E \in \mathcal{M}$ there is an $f \in L^+(\mu)$ such that $\nu(E) = \int_E f \, d\mu$. If $\mu$ is semifinite, then $f$ is unique up to a $\mu$-null set.

Thm 151: (Lebesgue Decomposition Theorem) Let $\mu, \nu$ be a measure on $\mathcal{M}$ with $\mu$ and $\nu$ $\sigma$-finite. Then there exist measures $\lambda, \rho$ on $\mathcal{M}$ such that $\rho \ll \mu$, $\lambda \perp \mu$ and $\nu = \rho + \lambda$. Moreover, this decomposition is unique, i.e., if $\nu = \overline{\rho} + \overline{\lambda}$ with $\overline{\rho} \ll \mu$ and $\overline{\lambda} \perp \mu$, then $\overline{\rho} = \rho$ and $\overline{\lambda} = \lambda$.

Thm 152: Let $\mu, \nu$ be measures on $\mathcal{M}$, with $\mu$ a $\sigma$-finite measure. Then
\[
\nu = \nu_{ac} + \nu_s,
\]
with $\nu_{ac} \ll \mu$ given in Lemma 144 and
\[
\nu_s(E) := \sup \{ \nu(F) : F \in \mathcal{M}, F \subseteq E \text{ and } \mu(F) = 0 \}.
\]

Mike Janssen
5.2 Signed Measures

DEF 153: A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

(a) $\nu(\emptyset) = 0$

(b) $\nu$ takes at most one of the values $+\infty$ or $-\infty$; i.e., $\nu : \mathcal{M} \to (-\infty, +\infty]$ or $\nu : \mathcal{M} \to [-\infty, +\infty)$

(c) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ are mutually disjoint, then $\nu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \nu(E_n)$, and the series converges absolutely if $|\nu(\bigcup_{n=1}^{N} E_n)| < \infty$.

Rmrk: Positive measures are signed measures.

Rmrk: Condition (c) can be simplified to just countable additivity, using a rearrangement of $\{E_n\}_{n=1}^{\infty}$.

EXAMP 154:

(a) If $\mu_1, \mu_2$ are positive measures with at least one finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.

(b) If $f$ is a measurable function, either $\int_X f^+ \, d\mu$ or $\int_X f^- \, d\mu$ is finite, then

$$\nu(E) = \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

is a signed measure. Such $f$ are called extended $\mu$-integrable.

PROP 155: Let $\nu$ be a signed measure on $\mathcal{M}$. (Continuity from below) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$ then

$$\lim_{n \to \infty} \nu(E_n) = \nu(\bigcup_{n=1}^{\infty} E_n).$$

(Continuity from above) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ with $E_1 \supseteq E_2 \supseteq \cdots$ and $|\nu(E_1)| < \infty$ then

$$\lim_{n \to \infty} \nu(E_n) = \nu(\bigcap_{n=1}^{\infty} E_n).$$

DEF 156: Let $\nu$ be a signed measure. A set $E \in \mathcal{M}$ is called

(a) positive if $\nu(F) \geq 0$ for all $F \in \mathcal{M}$ with $F \subseteq E$;

(b) negative if $\nu(F) \leq 0$ for all $F \in \mathcal{M}$ such that $F \subseteq E$;

(c) null if $\nu(F) = 0$ for all $F \in \mathcal{M}$ such that $F \subseteq E$.

EXAMP 157: Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function in $L^1(m)$. Further suppose $f(x) > 0$ for $x > 0$. Define $\nu$ by

$$\nu(E) := \int_E f \, dm$$

for all $E \in \mathcal{B}_\mathbb{R}$. Then $E \in \mathcal{B}_\mathbb{R}$ is a positive set if $E \subseteq [0, \infty)$, except possibly some $m$-null set; $E \in \mathcal{B}_\mathbb{R}$ is a negative set if $E \subseteq (-\infty, 0]$, except possibly some $m$-null set; and $E$ is a null set if $m(E) = 0$.

PROP 158: Let $\nu$ be a signed measure on $\mathcal{M}$. Let $E \in \mathcal{M}$ be such that $0 < \nu(E) < \infty$. Then there is a positive subset $F$ of $E$ such that $\nu(F) > 0$.

THM 159: (Hahn Decomposition Theorem) Let $\nu$ be a signed measure on $\mathcal{M}$. Then there are sets $P, N \in \mathcal{M}$ such that $P \cap N = \emptyset$, $P \cup N = X$, $P$ is positive, and $N$ is negative.

DEF 160: A pair of sets $P$ and $N$ satisfying the conclusion of Theorem 159 is called a Hahn decomposition for $\nu$.

THM 161: (Jordan Decomposition Theorem) Let $\nu$ be a signed measure on $\mathcal{M}$. There are unique measures $\nu^+$ and $\nu^-$ on $\mathcal{M}$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \subseteq \nu^-$. The decomposition in Theorem 161 is called the Jordan decomposition of $\nu$. The measure $\nu^+$ is called the positive (upper) variation and the measure $\nu^-$ is called the negative (lower) variation. The positive measure $|\nu| = \nu^+ + \nu^-$ is called the total variation.

DEF 162: Let $\nu, \omega$ be signed measures on $\mathcal{M}$.

(a) $E \in \mathcal{M}$ is $\sigma$-finite (with respect to $\nu$) if it is $\sigma$-finite for $|\nu|$.

(b) $\nu$ is $\sigma$-finite if $|\nu|$ is.

(c) $\nu$ and $\omega$ are mutually singular if $|\nu| \perp |\omega|$.

(d) $\nu$ is absolutely continuous with respect to $\omega$ if $|\nu| \ll |\omega|$.

Mike Janssen
Def 164: Suppose that $\nu$ is a signed measure on $\mathcal{M}$. Set $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$, and define
\[
\int_X f \, d\nu = \int_X f \, d\nu^+ - \int_X f \, d\nu^-
\]
for all $f \in L^1(\nu)$.

Thm 165: (Lebesgue-Radon-Nikodym Theorem) Let $\nu$ be a $\sigma$-finite signed measure on $\mathcal{M}$ and let $\mu$ be a positive $\sigma$-finite measure on $\mathcal{M}$. Then there are unique $\sigma$-finite signed measures $\lambda, \rho$ on $\mathcal{M}$ such that
\[
\lambda \perp \mu, \rho \ll \mu, \quad \text{and} \quad \nu = \rho + \lambda.
\]
Moreover, there is an extended $\mu$-integrable function $f$ such that $d\rho = f \, d\mu$. The function $f$ is unique up to a $\mu$-null set. The decomposition of $\nu$ into $\lambda$ and $\rho$ is unique.

Example 166: \[\text{(a) Let $\nu$ be a $\sigma$-finite signed measure. Let $P, N$ be a Hahn decomposition for $\nu$. Then $\nu(E) = \nu^+(E) - \nu^-(E) = \int_E \chi_P \, d\nu^+ - \int_E \chi_N \, d\nu^- = \int_E [\chi_P - \chi_N] \, d|\nu|$. Thus $\nu \ll |\nu|$ and $\frac{d\nu}{d|\nu|} = \chi_P - \chi_N.$}
\]
\[\text{(b) Let $F: \mathbb{R} \to \mathbb{R}$ be given by}
\]
\[F(x) = \begin{cases} 0, & x < 0 \\ 3 - e^{-x}, & 0 \leq x < 1 \\ 4 - e^{-x}, & 1 \leq x < \infty. \end{cases}
\]

Then $F$ is nondecreasing and right-continuous. The Lebesgue-Stieltjes measure $\mu_F$ satisfies $\mu_F(E) = \int_E e^{-x} \, d\mu + 2\delta_0 + \delta_1$.

Example 167: \[\text{(a) If $\psi$ is the Cantor-Vitali function then set}
\]
\[F(x) = \begin{cases} 0, & x < 0 \\ \psi(x), & 0 \leq x < 1 \\ 1, & x > 1 \end{cases}
\]

which is a continuous, nondecreasing function. If $\mu_F$ is the associated Lebesgue-Stieltjes measure, then it can be shown that $\mu_F \perp m$.

\[\text{(b) The measure $\mu: \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by $\mu(E) := \text{card}(E \cap \mathbb{Z})$ is singular with respect to $m$. The measure $\mu$ is discrete.}\]

6 Differentiation

We will look at using the LRN Theorem in particular contexts. We first consider the setting where $(X, \mathcal{M}) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. In this setting, there are ways of identifying $\frac{d\nu}{dm}$ more explicitly.

Def 168: A family $\{E_r\}_{r > 0} \subseteq \mathcal{B}_{\mathbb{R}^n}$ is said to shrink nicely to $x \in \mathbb{R}^n$ if
\[\text{(a) $E_r \subseteq B(r, x)$ for each $r > 0$}
\]
\[\text{(b) There exists $\alpha > 0$ such that $m(E_r) \geq \alpha m(B(r, x))$ for all $r > 0$.}\]

Def 169: \[\text{A measurable function $f: \mathbb{R}^n \to \mathbb{R}$ is locally integrable with respect to $m$ if $\int_K |f| \, dm < \infty$ for every compact set $K \in \mathcal{B}_{\mathbb{R}^n}$. The space of locally integrable functions on $\mathbb{R}^n$ is called $L^1_{\text{loc}}(\mathbb{R})$, or just $L^1_{\text{loc}}$.}\]

Mike Janssen
Def 170: Let \( f \in L_{\text{loc}}^1 \) be given. For each bounded set \( E \subseteq \mathbb{R}^n \) with \( m(E) > 0 \) we define the mean value (or average value) of \( f \) over \( E \) by

\[
\frac{1}{m(E)} \int_E f \, dm.
\]

Notation: \( \int f \, dm = \frac{1}{m(E)} \int_E f \, dm \) (assuming \( 0 < m(E) < \infty \))

Thm 171: (Lebesgue Differentiation Theorem) Suppose that \( f \in L_{\text{loc}}^1 \). For each Lebesgue point \( x \in \mathbb{R}^n \) of \( f \), we have

\[
\lim_{r \to 0^+} \int_{E_r} |f(y) - f(x)| \, dm(y) = 0
\]

and

\[
\lim_{r \to 0^+} \int_{E_r} f(y) \, dm(y) = f(x)
\]

for each \( \{E_r\}_{r>0} \subseteq \mathcal{B}(\mathbb{R}^n) \) that shrink nicely to \( x \). Recall that the elements of \( L^1 \) (and \( L_{\text{loc}}^1 \)) are actually equivalence classes of functions.

Thm 172: Let \( f \in L_{\text{loc}}^1 \) be given. Then for a.e. \( x \in \mathbb{R}^n \), \( \lim \int_{B(r,x)} f \, dm = f(x) \). This means that, given a representative \( g \) for the equivalence class of \( f \), we will find that \( \lim \int_{B(r,x)} g \, dm = f(x) \) for a.e. \( x \in \mathbb{R}^n \). This limit is independent of the choice of the equivalence class representative. The function

\[
f^*(x) := \begin{cases} 
\lim_{r \to 0^+} \int_{B(r,x)} f \, dm & \text{if the limit exists} \\
0 & \text{otherwise}
\end{cases}
\]

is called the precise representative for the equivalence class \( f \in L_{\text{loc}}^1 \).

Def 173: For each \( f \in L_{\text{loc}}^1 \) (\( f \) is a representative of an equivalence class), the set

\[
L_f := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \int_{B(r,x)} |f(y) - f(x)| \, dm(y) = 0 \right\}
\]

is called the Lebesgue set of \( f \), and its points are called Lebesgue points of \( f \).

Thm 174: If \( f \in L_{\text{loc}}^1 \), then \( m(\mathbb{R}^n \setminus L_f) = 0 \).

Rmrk: Theorem 174 states that \( f \in L_{\text{loc}}^1 \) can be identified with the precise representative for a.e. \( x \in \mathbb{R}^n \). Theorem 171 (LD) says that we can replace \( B(r,x) \) with \( \{E_r\}_{r>0} \) that shrink nicely to \( x \).

Thm 175: If \( f \in L^1(m) \), then for each \( \varepsilon > 0 \) there exists a continuous \( g : \mathbb{R}^n \to \mathbb{R} \) such that \( ||f - g||_{L^1} < \varepsilon \).

Def 176: Let \( f \in L_{\text{loc}}^1 \) be given. The Hardy-Littlewood Maximal function \( Hf : \mathbb{R}^n \to [0, \infty] \) is given by

\[
Hf(x) = \sup_{r>0} \int_{B(r,x)} |f| \, dm.
\]

Prop 177: Let \( f, g \in L_{\text{loc}}^1 \) be given. Then

(i) \( 0 \leq Hf(x) \leq +\infty \) for all \( x \in \mathbb{R} \).

(ii) \( H(f + g) \leq Hf + Hg \).

(iii) \( H(cf) = |c|Hf \) for all \( c \in \mathbb{R} \).

(iv) \( Hf \) is lower semi-continuous.

(v) \( Hf \) is \( \mathcal{B}(\mathbb{R}^n) \)-measurable.

Rmrk: Recall that a function \( h : \mathbb{R}^n \to \mathbb{R} \) is lower semi-continuous if the set \( \{x \in \mathbb{R}^n : h(x) > a\} \) is open for each \( a \in \mathbb{R} \). Equivalently, \( h \) is lower semi-continuous if \( \liminf_{y \to x} h(y) \geq h(x) \).

Mike Janssen
Example 178: \(^{47}\) Consider \(\chi_{[0,1]} \in L^1_{\text{loc}}\). (In fact, \(\chi_{[0,1]} \in L^1\) ) Then
\[
H \chi_{[0,1]} = \sup_{r>0} \frac{\int_{(x-r,x+r)} \chi_{[0,1]} \, dm}{2r} = \sup_{r>0} \frac{1}{2r} m([0,1] \cap (x-r,x+r)) = \begin{cases} \frac{1}{2(1-x)} & \text{if } x \leq 0 \\ 1 & \text{if } x \in (0,1) \\ \frac{1}{2x} & \text{if } x \geq 1. \end{cases}
\]

Observe that \(\chi_{[0,1]} \in L^1\) but \(H \chi_{[0,1]} \notin L^1\). In general, \(Hf \notin L^1(\mathbb{R})\) unless \(f = 0\) for a.e. \(x \in \mathbb{R}^n\).

Thm 179: (Chebyshev’s Inequality): Let \((X, \mathcal{M}, \mu)\) be a measure space. If \(f \in L^p(\mu)\) for some \(p \in [1, \infty)\), then for all \(\alpha > 0\), \(\mu(\{x \in X : |f(x)| > \alpha\}) \leq \frac{1}{\alpha^p} \|f\|_{L^p}^p\).

Def 180: Let \((X, \mathcal{M}, \mu)\) be a measure space. For a measurable function \(f : X \to \overline{\mathbb{R}}\), define
\[
[f]_p := \sup_{\alpha > 0} \alpha \mu(\{x \in X : |f(x)| > \alpha\})
\]

for \(p \in [1, \infty)\). We say that \(f\) is in weak-\(L^p\) if \([f]_p < \infty\).

Rmk: \([\cdot]_p\) is not a norm, as it does not satisfy the triangle inequality.

Rmk: \(L^p \subseteq\) weak-\(L^p\) and \([f]_p \leq \|f\|_p\). Although \(Hf \notin L^1\) in general, it will belong to weak-\(L^1\) if \(f \in L^1\).

Thm 181: (Maximal Theorem) There is a constant \(c > 0\) such that for each \(f \in L^1\) and all \(\alpha > 0\) such that for each \(f \in L^1\) and all \(\alpha > 0\), \(m(\{x \in X : Hf(x) > \alpha\}) \leq \frac{c}{\alpha^p} \|f\|_{L^1}^p\).

Lma 182: (Simple Vitali Covering Lemma) Let \(C\) be a collection of open balls in \(\mathbb{R}^n\) and set \(U := \bigcup_{B \in C} B\). If \(\tau < m(U)\) then there exists disjoint balls \(\{B_j\}_{j=1}^k \subseteq C\) such that \(\sum_{j=1}^k m(B_j) > \frac{\tau}{3^k}\).

Def 183: \(^{48}\) A signed measure \(\nu\) on \(\mathcal{B}_{\mathbb{R}^n}\) is regular if

(a) \(|\nu|(K) < \infty\) for each compact \(K \subseteq \mathbb{R}^n\).

(b) \(|\nu|(E) = \inf \{\nu(U) : U \text{ is open and } E \subseteq U\}\) for all \(E \in \mathcal{B}_{\mathbb{R}^n}\).

Rmk: If \(\nu\) is positive, then Definition 183 is equivalent to Definition 114.

Rmk: If \(\nu\) is regular, then \(|\nu|\) is a Radon measure.

Rmk: If \(\nu\) is regular, then \(\nu\) is \(\sigma\)-finite.

Rmk: If \(d\nu = f \, dm\) then \(\nu\) is regular iff \(|f| \in L^1_{\text{loc}}\).

Thm 184: Let \(\nu\) be a regular signed measure on \(\mathcal{B}_{\mathbb{R}^n}\). Let \(d\nu = d\lambda + f \, dm\) be its Lebesgue-Radon-Nikodym decomposition with respect to \(m\). Then for \(m\)-a.e. \(x \in \mathbb{R}^n\),
\[
\lim_{r \to 0^+} \frac{\nu(E_r)}{m(E_r)} = f(x)
\]
for every family \(\{E_r\}_{r>0} \subseteq \mathcal{B}_{\mathbb{R}^n}\) that shrinks nicely to \(x\).

Note: \(\nu\) regular implies \(\nu\) \(\sigma\)-finite implies the Lebesgue-Radon-Nikodym Theorem can be used.

Thm 185: \(^{49}\) (Lusin’s Theorem) Let \(f : \mathbb{R}^n \to \mathbb{R}\) be a Borel-measurable function. Suppose there is an \(A \in \mathcal{B}_{\mathbb{R}^n}\) such that \(m(A) < \infty\) and \(f(x) = 0\) for all \(x \in A^c\). For each \(\varepsilon > 0\) there exists continuous \(g : \mathbb{R}^n \to \mathbb{R}\) such that \(\sup_{x \in \mathbb{R}^n} |g(x)| \leq \sup_{x \in \mathbb{R}^n} |f(x)|\) and \(m(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \varepsilon\).

6.1 Functions of Bounded Variation

Thm 186: \(^{50}\) (Differentiability of Monotone Functions) Let \(F : \mathbb{R} \to \mathbb{R}\) be a nondecreasing function, and define \(G(x) = F(x^+)\).

(a) The points of discontinuity for \(F\) constitute an \(m\)-null set. In fact, the set is countable.
(b) \( F \) is differentiable at a.e. \( x \in \mathbb{R} \) and \( F' = G' \) a.e. in \( \mathbb{R} \).

**Def 187:** Let \( F : \mathbb{R} \to \mathbb{R} \) be given. Define \( T_F : \mathbb{R} \to [0, \infty] \) by

\[
T_F(x) = \sup \left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N} \text{ and } -\infty < x_0 < x_1 < \cdots < x_n = x \right\}.
\]

We call \( T_F \) the **total variation function of** \( F \). If \( [a, b] \subseteq \mathbb{R} \), then \( T_F(b) - T_F(a) \) is the **total variation of** \( F \) over \([a, b]\).

**Def 188:** Let \( F : \mathbb{R} \to \mathbb{R} \) be given. If \( T_F(\mathbb{R}) < \infty \) then we say that \( F \) is of **bounded variation on** \( \mathbb{R} \). We set

\[
BV(\mathbb{R}) := \{ F : \mathbb{R} \to \mathbb{R} : T_F(\mathbb{R}) < \infty \}.
\]

**Def 189:** Let \( F : [a, b] \to \mathbb{R} \) be given. If \( T_F(b) - T_F(a) < \infty \) then we say that \( F \) is of **bounded variation over** \([a, b]\). We set

\[
BV([a, b]) := \{ F : \mathbb{R} \to \mathbb{R} : T_F(b) - T_F(a) < \infty \}.
\]

**Rmrk:** \( T_F \) is increasing.

**Rmrk:** If \( F \in BV(\mathbb{R}) \), then \( T_F(b) - T_F(a) = \sup \left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N} \text{ and } a = x_0 < \cdots < x_n = b \right\} \).

**Rmrk:** \( BV(\mathbb{R}) \) and \( BV([a, b]) \) are vector spaces.

**Rmrk:** If \( F \in BV([a, b]) \) then

\[
F(x) := \begin{cases} 
F(a) & x \leq a \\
F(x) & x \in [a, b] \in BV(\mathbb{R}). \\
F(b) & x \geq b
\end{cases}
\]

**Example 190:**

(a) All bounded monotone functions are \( BV(X) \) for \( X \in \{\mathbb{R}, [a, b]\} \).

(b) The function \( \sin(x) \) is not in \( BV(\mathbb{R}) \), but \( \sin(x) \cdot \chi_K \in BV(\mathbb{R}) \) for any compact \( K \subseteq \mathbb{R} \).

(c) The function

\[
F(x) = \begin{cases} 
x^k \sin(1/x) & x > 0 \\
0 & x \leq 0
\end{cases}
\]

is not in \( BV \) if \( k = 0, 1 \).

**Lma 191:** If \( F \in BV(\mathbb{R}) \), then \( x \mapsto T_F(x) + F(x) \) and \( x \mapsto T_F(x) - F(x) \) are both nondecreasing.

**Thm 192:** If \( F \in BV(\mathbb{R}) \), then \( F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F) \), and \( \frac{1}{2}(T_F + F) \) and \( \frac{1}{2}(T_F - F) \) are nondecreasing. Also, if \( F \) is the difference of 2 nondecreasing bounded functions, then \( F \in BV(\mathbb{R}) \).

**Thm 193:**

(a) If \( F \in BV(\mathbb{R}) \), then \( F(x^+) \) and \( F(x^-) \) exist for all \( x \in \mathbb{R} \), and \( F(-\infty) \) and \( F(+\infty) \) exist as well.

(b) If \( F \in BV(\mathbb{R}) \), then \( F \) is discontinuous on a set of at most countably many points.

(c) If \( F \in BV(\mathbb{R}) \) and \( G(x) := F(x^+) \) then \( F' \) exists a.e. and \( F' = G' \) a.e. in \( \mathbb{R} \).

**Def 194:** If \( F \in BV(\mathbb{R}) \), then the decomposition \( F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F) \) is called the **Jordan decomposition** of \( F \). Moreover, \( \frac{1}{2}(T_F + F) \) is called the **positive variation of** \( F \) and \( \frac{1}{2}(T_F - F) \) is called the **negative variation of** \( F \).

**Def 195:** Set \( \text{NBV}(\mathbb{R}) := \{ F \in BV(\mathbb{R}) : F(-\infty) = 0 \) and \( F \) is right-continuous \}.

**Lma 196:** If \( F \in BV(\mathbb{R}) \) then \( T_F(-\infty) = 0 \). If \( F \in BV(\mathbb{R}) \) is right-continuous, then \( T_F \in \text{NBV}(\mathbb{R}) \).

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Mike Janssen
Thm 197: If \( \mu \) is a finite signed Borel measure and \( F : \mathbb{R} \to \mathbb{R} \) is defined by \( F(x) = \mu((-\infty, x]) \) for all \( x \in \mathbb{R} \) then \( F \in \text{NBV} \). If \( F \in \text{NBV} \) then there is a unique signed Borel measure \( \mu_F \) such that \( \mu_F((-\infty, x]) = F(x) \) for all \( x \in \mathbb{R} \). Moreover, \( |\mu_F| = \mu_{TF} \); i.e., \( |\mu_F|((-\infty, x]) = T_F(x) \) for all \( x \in \mathbb{R} \).

Prop 198: \(^{53}\) Let \( F \in \text{BV} \) be right-continuous. Then \( F' \in L^1 \). Moreover, if \( F \in \text{NBV} \) then

- (a) \( \mu_F \perp m \Leftrightarrow F' = 0 \) for m-a.e. \( x \in \mathbb{R} \).
- (b) \( \mu_F \ll m \Leftrightarrow F(x) = \int_{(-\infty, x]} F' \, dm \) for all \( x \in \mathbb{R} \).

Def 199: We say that \( F : \mathbb{R} \to \mathbb{R} \) is absolutely continuous if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( \{(a_j, b_j)\}_{j=1}^N \) are disjoint intervals with \( \sum_{j=1}^N b_j - a_j < \delta \) we also have \( \sum_{j=1}^N |F(b_j) - F(a_j)| < \varepsilon \). We say \( F \) is absolutely continuous on \([a, b]\) if the above property holds with intervals taken as subsets of \([a, b]\).

Prop 200: If \( F \in \text{NBV} \), then \( F \) is absolutely continuous if and only if \( \mu_F \ll m \).

Cor 201: \(^{54}\) If \( f \in L^1(m) \), then \( F(x) := \int_{(-\infty, x]} f \, dm \) in \( \text{NBV} \) is absolutely continuous and \( F' = f \) a.e. Also, if \( F \in \text{NBV} \) is absolutely continuous then \( F' \in L^1 \) and \( F(x) = \int_{(-\infty, x]} F' \, dm \).

Thm 202: (FTC for Lebesgue Integrals) Let \([a, b] \subseteq \mathbb{R} \) and \( F : [a, b] \to \mathbb{R} \) be given. TFAE:

- (a) \( F \) is absolutely continuous on \([a, b]\)
- (b) \( F(x) = F(a) + \int_{[a,x]} f \, dm \) for some \( f \in L^1([a,b], m) \)
- (c) \( F' \) exists a.e. in \([a, b]\), \( F' \in L^1([a,b], m) \) and \( F(x) = F(a) + \int_{[a,x]} F' \, dm \).

Def 203: If \( F \in \text{NBV} \), then the integral of a Borel-measurable function with respect to \( \mu_F \) is called a Lebesgue-Stieltjes integral (if it makes sense; see Definition 164). We may denote the integral by \( \int g \, dF \) or \( \int g(x) \, dF(x) \).

If \( F \) is absolutely continuous, then \( \int g \, dF = \int gF' \, dm \).

Thm 204: Suppose \( F, G \in \text{NBV} \), and at least one of \( F \) or \( G \) is continuous. Then for any \((a,b) \subset \mathbb{R} \), we have

\[
\int_{[a,b]} F \, dG = [F(b)G(b) - F(a)G(a)] - \int_{[a,b]} G \, dF.
\]

7 Change of Variables

Prop 205: \(^{55}\) Let \((X, \mathcal{M}, \mu)\) be a measure space and \((Y, \mathcal{N})\) a measurable space. Let \( T : X \to Y \) be measurable, and define \( \mu \circ T^{-1} : \mathcal{N} \to [0, \infty] \) by \( (\mu \circ T^{-1})(E) = \mu(T^{-1}(E)) \). Then \( \mu \circ T^{-1} \) is a measure on \( \mathcal{N} \).

Def 206: The measure \( \mu \circ T^{-1} \) is called the push forward of \( \mu \) through \( T \), or the measure induced by \( \mu \) and \( T \).

Example 207: Let \( T : [0, 2\pi) \to \mathbb{R}^2 \) be given by \( T(t) = (\cos(t), \sin(t)) \). Then \( T \) is a bijection between \([0, 2\pi)\) and \( S^1 \). Let \( \mathcal{B}_S \) be the \( \sigma \)-algebra generated by \( T^{-1} : S^1 \to [0, 2\pi) \) (cf. Definition 23). Then \( T \) is measurable. The surface Lebesgue measure on \( S^1 \) is \( m \circ T^{-1} \) (the push forward of \( m \) through \( T \)). If \( E \in \mathcal{B}_S \), then \( (m \circ T^{-1})(E) \) gives the “total angle” subtended by \( E \).

Thm 208: (General Change of Variables) Let \((X, \mathcal{M}, \mu)\) be a measure space and \((Y, \mathcal{N})\) be a measurable space. Suppose that \( T : X \to Y \) is a measurable function. Then for any \( \mathcal{N} \)-measurable functions \( f : Y \to \mathbb{R} \), we have

\[
\int_X f(T(x)) \, d\mu = \int_Y f(y) \, d(\mu \circ T^{-1})
\]

provided one of the integrals makes sense.

Cor 209: Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces, and let \( T : X \to Y \) be measurable. If \( \nu \ll \mu \circ T^{-1} \), then there is a measurable \( \varphi : Y \to \mathbb{R} \) such that for each \( f \in L^1(\nu) \) satisfying \( f \circ T \in L^1(\mu) \), we have

\[
\int_E f(y) \, d\nu(y) = \int_{T^{-1}(E)} f(T(x)) \varphi(T(x)) \, d\mu(x)
\]

Mike Janssen
for all $E \in \mathcal{M}$.

**Thm 210:** Suppose that $T : [a, b] \to [c, d]$ is an increasing, absolutely continuous bijection. Let $f \in L^1([c, d], m)$ be given such that $f \circ T \in L^1([a, b])$. Then

$$\int_{[a, b]} f(T(x))T'(x) \, d m(x) = \int_{[c, d]} f(y) \, d m(y).$$

**Thm 211:** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear transformation. If $f \in L^1(m)$, then $f \circ T \in L^1(m)$ and

$$\int_{\mathbb{R}^n} f \, d m = |\det T| \int_{\mathbb{R}^n} f \circ T \, d m.$$

**Notation:** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. If $G : \Omega \to \mathbb{R}^n$ has continuously differentiable components, i.e., $G = (G_1, \cdots, G_n)$ with $G_1, \cdots, G_n \in C^1(\Omega)$. Then $D_x G : \Omega \to \mathbb{R}^{n \times n}$ is given by $[D_x G]_{ij} := \partial G_i / \partial x_j$. We say $G$ is a $C^1$-diffeomorphism if $G$ is bijective and $G$ and $G^{-1}$ are both continuously differentiable.

**Thm 212:** (Change of Variables) Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and that $G : \Omega \to \mathbb{R}^n$ is a $C^1$ diffeomorphism.

(a) If $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(y) \, d m(y) = \int_{\Omega} f(G(x)) |\det D_x G(x)| \, d m(x).$$

(b) If $E \subseteq \Omega$ and $E \subseteq \mathcal{L}^n$ then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E |\det D_x G(x)| \, d m(x)$.

**Integration in Polar Coordinates**

Set $S^{n-1} := \{ x \in \mathbb{R}^n : ||x|| = 1 \}$. For $x \in \mathbb{R}^n \setminus \{0\}$, define $r(x) = ||x||$ and $\theta(x) = x/||x||$. It can be verified that the map $G : (0, \infty) \times S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ given by $G(r, \theta) = r \theta$ is a $C^1$-diffeomorphism. Since $G$ is a bijection and continuous we may define a measure $m_* \mapsto \mathcal{B}(0, \infty) \times S^{n-1}$ by $m_*(E) = m(G(E))$. Define the measure $\rho_n$ on $\mathcal{B}(0, \infty)$ by $\rho_n(E) = \int_E r^{n-1} \, d m(r)$.

**Thm 213:** There is a unique Borel measure $\sigma_{n-1}$ on $S^{n-1}$ such that $m_* = \rho_n \times \sigma_{n-1}$.

### 8 Dual Spaces

**Def 214:** If $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a normed vector space, the space of continuous linear functionals $(L(\mathcal{X}, \mathbb{R}), \|\cdot\|)$ is called the (continuous) **dual space** of $\mathcal{X}$. It is denoted by $\mathcal{X}^*$ or $(\mathcal{X}, \|\cdot\|_{\mathcal{X}^*})$.

**Rmrk:** $\mathcal{X}^*$ is always a $B$-space.

**Example 215:** Suppose that $\mathcal{X} = \mathbb{R}^n$ and $\|\cdot\| = \sqrt{x \cdot x}$. For each $\varphi \in \mathcal{X}^*$ there is a unique $y \in \mathbb{R}^n$ such that $\varphi(x) = y \cdot x$ for all $x \in \mathbb{R}^n$. Clearly the functional $x \mapsto y \cdot x$ is in $\mathcal{X}^*$. Thus, $\mathcal{X}^*$ can be identified with $\mathcal{X}$ itself. It can be shown that $\mathcal{X}$ and $\mathcal{X}^*$ are isometric.

**Thm 216:** (Riesz Representation Theorem for $L^p$ spaces) Suppose that $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$. Then for each $\varphi \in (L^q)^*$ there is a $g \in L^q$ such that $\varphi(f) = \int_X f \, d \mu$ for all $f \in L^p$. If $\mu$ is $\sigma$-finite, then the result holds when $p = 1$. (In which case $q = \infty$).

**Rmrk:** It will be shown (later) that $(L^p)^* \cong L^q$.

**Thm 217:** Suppose $p, q \in [1, \infty]$ satisfy $1/p + 1/q = 1$, and that $g : X \to \mathbb{R}$ satisfies

(i) $fg \in L^1$ for all $f \in \Sigma := \{ h \in L^1 : h$ is simple $\}$.

(ii) $M_q(g) := \sup \left\{ \int_X f g \, d \mu : f \in \Sigma \cap L^p$ and $||f||_{L^p} \leq 1 \right\} < \infty$.

(iii) Either $S_g := \{ x \in X : |g(x)| \neq 0 \}$ is $\sigma$-finite or $\mu$ is semi-finite.

![Mike Janssen](image)
Then $g \in L^q$. Moreover, $\|g\|_{L^q} = M_q(g)$.

Cor 218: If $p \in (1, \infty)$, then $L^p$ is reflexive; i.e., $(L^p)^{**} \cong L^p$.

Def 219: A **directed set** $A$ is a non-empty set with a relation $\preceq$ such that $\alpha \preceq \alpha$ for all $\alpha \in A$; if $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then $\alpha \preceq \gamma$; if $\alpha, \beta \in A$ then there is $\gamma \in A$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. An element of a directed set is called an index.

**Example 220:**

(a) Any nonempty subset of $\mathbb{R}$ with the usual ordering is a directed set.

(b) Let $\mathcal{B}$ be a neighborhood basis for a topology on $X$; if $\mathcal{B} \subseteq \mathcal{T}$ and for each $x \in X$ there is $\mathcal{N} \subseteq \mathcal{B}$ such that $x \in V$ for all $V \in \mathcal{N}$ and, if $U \in \mathcal{T}$ and $x \in U$ there is $V \in \mathcal{N}$ such that $V \subseteq U$. For each $x \in X$, set $\mathcal{N}_x := \{U \in \mathcal{B} : x \in U\}$. Fix $x \in X$. If we say that $U \preceq V$ whenever $U, V \in \mathcal{N}_x$ and $U \supseteq V$ then $\mathcal{N}_x$ is a directed set.

Def 221: A **net** in $X$ is a function $x : A \to X$ with $A$ a directed set. We denote the function by $\langle x_\alpha \rangle_{\alpha \in A}$. The set $A$ is called the **index set**.

Def 222: Let $(X, T)$ be a topological space and let $E \subseteq X$ be given. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net. We say

(a) $\langle x_\alpha \rangle$ is **eventually in** $E$ if there is an $\alpha_0 \in A$ such that $x_\alpha \in E$ whenever $\alpha_0 \preceq \alpha$.

(b) $\langle x_\alpha \rangle$ is **frequently in** $E$ if for all $\alpha \in A$ there is a $\beta \in A$ such that $\alpha \preceq \beta$ and $x_\beta \in E$.

(c) $\langle x_\alpha \rangle$ **converges to a point** $x \in X$ if for every $U \in T$ with $x \in U$, $\langle x_\alpha \rangle$ is eventually in $U$. For brevity, we write $x_\alpha \to x$.

Prop 223: If $(X, T)$ and $(Y, S)$ are topological spaces and $f : X \to Y$ then $f$ is continuous at $x \in X$ iff for each net $\langle x_\alpha \rangle$ converges to $x$ we have $\langle f(x_\alpha) \rangle$ converges to $f(x)$.

Def 224: Let $X$ be a set and $(\{Y_\alpha, T_\alpha\})_{\alpha \in A}$ a family of topological spaces. Given a family of mappings $\{f_\alpha : X \to Y_\alpha\}_{\alpha \in A}$ there is a unique weakest topology on $X$ that makes each $f_\alpha$ continuous. This topology is called the **weak topology** generated by $f_\alpha$.

Def 225: Let $(\mathcal{X}, \| \cdot \|)$ be a normed vector space. The **weak topology** on $\mathcal{X}$ is the weak topology generated by $\mathcal{X}^*$. Convergence in this topology is called **weak convergence**.

Rmrk: If $\langle x_\alpha \rangle \subseteq \mathcal{X}$ is a net, we say $x_\alpha \to x$ strongly iff for each $\varepsilon > 0$, eventually $\|x_\alpha - x\| < \varepsilon$. On the other hand, $x_\alpha \to x$ **weakly** (written $x_\alpha \rightharpoonup x$) iff $f(x_\alpha) \to f(x)$ for all $f \in \mathcal{X}^*$. Recall that $\mathcal{X}$ can be identified as a subset of $\mathcal{X}^{**}$ so each $x \in \mathcal{X}$ can be identified with a linear functional on $\mathcal{X}^*$. Weak convergence is called **weak* convergence.**

Rmk: If $\langle f_\alpha \rangle \subseteq \mathcal{X}^*$ is a net, then $f_\alpha \to f$ weak* (written $f_\alpha \rightharpoonup f$) if $f_\alpha(x) \to f(x)$ for all $x \in \mathcal{X}^*$. Def 226: Let $(\mathcal{X}, \| \cdot \|)$ be a normed vector space. The **weak** topology on $\mathcal{X}$ is the weak topology generated by $\mathcal{X}^*$ (identified as a subset of $\mathcal{X}^{**}$). Convergence in the weak topology is called **weak convergence**.

Rmk: If $\langle f_\alpha \rangle \subseteq \mathcal{X}^*$ is a net, then $f_\alpha \to f$ weak* (written $f_\alpha \rightharpoonup f$) if $f_\alpha(x) \to f(x)$ for all $x \in \mathcal{X}^*$. Def 227: A **subnet** of a net $\langle x_\alpha \rangle_{\alpha \in A}$ is a net $\langle y_\beta \rangle_{\beta \in B}$ together with a map $\beta \to \alpha_\beta$ from $B$ to $A$ such that for each $\alpha_0 \in A$ there is a $\beta_0 \in B$ such that $\alpha_0 \preceq \alpha_\beta$ for all $\beta \in B$ with $\beta_0 \preceq \beta$. Moreover, we require that $y_\beta = x_{\alpha_\beta}$.

Rmrk: **Convergent nets** in $\mathcal{X}^*$ must have a weak* convergent subnet.

Cor 228: (to Riesz’s Theorem) If $\mu$ is $\sigma$-finite and $\langle f_\alpha \rangle$ is a bounded net in $L^\infty$. Then there is a subnet $\langle g_\beta \rangle$ of $\langle f_\alpha \rangle$ and a function $g \in L^\infty$ such that $g_\beta \rightharpoonup g$ in $L^\infty$ ; i.e.,

$$\int_X g_\beta f \, d\mu \to \int_X g f \, d\mu \quad \forall f \in L^1.$$  

Cor 229: (to Riesz’s Theorem) If $p \in (1, \infty)$, $q = \frac{p}{p-1}$ and $\langle f_\alpha \rangle$ is a bounded net in $L^p$ then there exists a subnet $\langle g_\beta \rangle$ of $\langle f_\alpha \rangle$ and a $g \in L^p$ such that $g_\beta \rightharpoonup g$ in $L^p$ ; i.e.,

$$\int_X g_\beta f \, d\mu \to \int_X g f \, d\mu \quad \forall f \in L^q.$$  

Cor 230: If $p \in (1, \infty)$ and $\{f_n\}_{n=1}^\infty$ is bounded in $L^p$ then there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that $f_{n_k} \to f$ in $L^p$ for some $f \in L^p$.

Using the Uniform Boundedness Principle (5.13 in Folland), we also have

Cor 231: If $p \in (1, \infty)$ and $\{f_n\}_{n=1}^\infty \subseteq L^p$ such that $f_n \to f$ in $L^p$ then $\sup_{n \in \mathbb{N}} \|f_n\|_{L^p} < \infty$.

See notes for a good counterexample of when the previous Corollary fails when $p = 1$.

Mike Janssen
(a)
(b)
(c)
Notes