CHALLENGING PROBLEMS FOR CALCULUS STUDENTS

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1. INTRODUCTION

In what follows I will post some challenging problems for students who have had some calculus, preferably at least one calculus course. All problems require a proof. They are not easy but not impossible. I hope you will find them stimulating and challenging.

2. PROBLEMS

(1) Prove that

\[ e^\pi > \pi^e. \]  \hspace{1cm} (2.1)

*Hint*: Take the natural log of both sides and try to define a suitable function that has the essential properties that yield inequality 2.1.

(2) Note that \( \frac{1}{4} \neq \frac{1}{2} \); but \( \left( \frac{1}{4} \right)^{\frac{1}{2}} = \left( \frac{1}{2} \right)^{\frac{1}{2}} \). Prove that there exists infinitely many pairs of positive real numbers \( \alpha \) and \( \beta \) such that \( \alpha \neq \beta \); but \( \alpha^\alpha = \beta^\beta \). Also, find all such pairs.

*Hint*: Consider the function \( f(x) = x^x \) for \( x > 0 \). In particular, focus your attention on the interval \((0, 1]\). Proving the existence of such pairs is fairly easy. But finding all such pairs is not so easy. Although such solution pairs are well known in the literature, here is a neat way of finding them: look at an article written by Jeff Bomberger\(^1\), who was a freshman at UNL enrolled in my calculus courses 106 and 107, during the academic year 1991-92.

(3) Let \( a_0, a_1, \ldots, a_n \) be real numbers with the property that

\[ a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1} = 0. \]

Prove that the equation

\[ a_0 + a_1x + a_2x^2 + \ldots a_nx^n = 0 \]

\(^1\)Jeffrey Bomberger, On the solutions of \( a^a = b^b \), *Pi Mu Epsilon Journal*, **Volume 9**(9)(1993), 571-572.
has at least one solution in the interval (0, 1).

(4) Suppose that $f$ is a continuous function on $[0, 2]$ such that $f(0) = f(2)$. Show that there is a real number $\xi \in [1, 2]$ with $f(\xi) = f(\xi - 1)$.

(5) Suppose that $f : [0, 1] \rightarrow [0, 1]$ is a continuous function. Prove that $f$ has a fixed point in $[0, 1]$, i.e., there is at least one real number $x \in [0, 1]$ such that $f(x) = x$.

(6) The axes of two right circular cylinders of radius $a$ intersect at a right angle. Find the volume of the solid of intersection of the cylinders.

(7) Let $f$ be a real-valued function defined on $[0, \infty)$, with the properties: $f$ is continuous on $[0, \infty)$, $f(0) = 0$, $f'$ exists on $(0, \infty)$, and $f'$ is monotone increasing on $(0, \infty)$.

Let $g$ be the function given by: $g(x) = \frac{f(x)}{x}$ for $x \in (0, \infty)$.

a) Prove that $g$ is monotone increasing on $(0, \infty)$.

b) Prove that, if $f'(c) = 0$ for some $c > 0$, and if $f(x) \geq 0$, for all $x \geq 0$, then $f(x) = 0$ on the interval $[0, c]$.

(8) Evaluate the integral $\int \frac{1}{x^4 + 1} dx$.

$\text{Hint: write } x^4 + 1 = (x^2 + 1)^2 - 2x^2. \text{ Factorize and do a partial fraction decomposition.}$

(9) Determine whether the improper integral $\int_0^\infty \sin(x) \sin(x^2) dx$ is convergent or divergent.

$\text{Hint: the integral is convergent.}$

(10) Let $f$ be a real-valued function such that $f$, $f'$, and $f''$ are all continuous on $[0, 1]$. Consider the series $\sum_{k=1}^\infty f\left(\frac{1}{k}\right)$.

(a) Prove that if the series $\sum_{k=1}^\infty f\left(\frac{1}{k}\right)$ is convergent, then $f(0) = 0$ and $f'(0) = 0$.

(b) Conversely, show that if $f(0) = f'(0) = 0$, then the series $\sum_{k=1}^\infty f\left(\frac{1}{k}\right)$ is convergent.