Review

Vector Fields:

1. Plot the vector field $\mathbf{F}(x, y) = < x, y >$.

2. Let $f$ be a real function in three variables and $\mathbf{F} = < f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) >$ a vector field in three variables. For the following, decide whether the formula makes sense, if it does express it in terms of partial derivatives, $f, f_1, f_2$ and $f_3$ and write carefully whether is a scalar or vector quantity.

(a) $\nabla \cdot (\nabla f)$.

(b) $\nabla (\nabla f)$.

(c) $\nabla (\nabla \mathbf{F})$.

(d) $\nabla \cdot \mathbf{F}$.

(e) $\nabla \times (\nabla f)$.

(f) $\nabla \times (\nabla \mathbf{F})$.

(g) $\nabla \cdot (\nabla \times \mathbf{F})$.

Line Integrals. In the following $f(x, y, z)$ is a continuous function everywhere.

1. Let $C$ be an oriented curve in a three dimensional space. Assume that $C$ is described parametrically by $< x(t), y(t), z(t) >$ for $a \leq t \leq b$, and $x(t), y(t), z(t)$ are continuous with continuous first derivatives. How do you compute the line integral of $f(x, y, z)$ with respect to the arc length $(\int_C f(x, y, z)ds)$? (i.e. State the evaluation Theorem I)

2. Let $g(x, y, z) = e^x$. Give an interpretation for $\int_C g(x, y, z)ds$, where $C$ is the piece of the helix $< \cos(t), \sin(t), t >$, for $0 \leq t \leq 1$.

3. Is $\int_C f(x, y, z)ds = \int_C f(x, y, z)ds$?

4. If $f(x, y, z) = 1$, give the interpretation of $\int_C f(x, y, z)ds$.

5. Let $C$ be an oriented curve in a three dimensional space. Assume that $C$ is described parametrically by $< x(t), y(t), z(t) >$ for $a \leq t \leq b$, and $x(t), y(t), z(t)$ are continuous with continuous first derivatives. How do you compute the line integral of $f(x, y, z)$ with respect to $x$ $(\int_C f(x, y, z)dx)$? (i.e. state the evaluation Theorem II) What about $\int_C f(x, y, z)dy$ and $\int_C f(x, y, z)dz$?

6. Is $\int_C f(x, y, z)dx = \int_C f(x, y, z)dx$?

7. What is it $\int_C \mathbf{F} \cdot d\mathbf{r}$ in terms of line integrals if $\mathbf{F} = < f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) >$ is a vector field?

8. Give an interpretation of $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = mg$, where $m$ is a mass and $g$ is the gravity.
Conservative fields, for two dimensional vector fields: In the following let \( \mathbf{F}(x, y) = <M(x, y), N(x, y)> \) be a vector field with continuous partial derivatives over \( \mathbb{R}^2 \).

Mark as FALSE or TRUE. If the statement is false, write the correct version.

1. If \( \mathbf{F} \) is a conservative field then \( \int_C M(x, y) \, ds = 0 \) for all closed path \( C \).
2. If \( \mathbf{F} \) is a conservative field then \( \int_C \mathbf{F} \, d\mathbf{r} \) does not depend on the path \( C \) but just on the initial and final point.
3. If \( M_x(x, y) = N_y(x, y) = 0 \) then \( \mathbf{F} \) is conservative.
4. If \( \mathbf{F} \) is conservative then \( M_y(x, y) = N_x(x, y) = 0 \).
5. If \( \mathbf{F} \) is conservative then there exists a function in two variables such that \( \nabla \cdot f(x, y) = \mathbf{F} \).

Surface Integrals.

1. State the Evaluation Theorem for surface integrals with all the assumptions.
2. Find the surface area of the portion of the cone \( \sqrt{x^2 + y^2} \) under the plane \( z = 4 \).
3. Compute \( \iint_S z^2 \, dS \), where \( S \) is the portion of the cone \( z = \sqrt{x^2 + y^2} \) above the rectangle \( 0 \leq x \leq 2, -1 \leq y \leq 2 \).

Green’s Theorem.

1. State Green’s Theorem, with all the assumptions.
2. What is the orientation of the curve in Green’s Theorem.
3. Let \( \mathbf{F}(x, y) = <\frac{x}{y}, y> \). Can you apply Green’s theorem to compute \( \int_C \mathbf{F} \, d\mathbf{r} \), where \( C \) is the circle with center the origin and radius 3? Explain your answer.
4. Let \( \mathbf{F} = <y + e^{\sqrt{y}}, 2x + \cos(y^2)> \). Compute \( \int_C \mathbf{F} \cdot \mathbf{r} \), where \( C \) is the triangle with vertices \((-1, 0), (1, 0), (0, 2)\).

The Divergence Theorem

1. State The Divergence Theorem with all the assumptions.
2. Compute the flux of the vector field \( \mathbf{F} = <x^3, y^3, z^3> \) through the sphere of radius 3.

Stokes’ Theorem.

1. State Stokes’ Theorem with all the assumptions. How it is different from Green’s Theorem?
2. Use Stoke’s Theorem to compute \( \int_C \mathbf{F} \cdot \mathbf{r} \) where \( \mathbf{F}(x, y, z) = <e^x, e^{-x}, e^z> \) and \( C \) is the boundary of the part of the plane \( 2x + 2y + 2z = 2 \) that lies in the first octant.
3. Use Stoke’s Theorem to compute \( \int_C \mathbf{F} \cdot \mathbf{r} \) where \( \mathbf{F}(x, y, z) = <x + y^2, y + z^2, z + x^2> \) where \( C \) is the triangle with vertices \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).