The PageRank Algorithm and Web Search Engines

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What is PageRank?

PageRank is an algorithm for ranking the importance of webpages.

It was developed in the late ’90’s by Larry Page and Sergey Brin, at that time grad students at Stanford.
References

Brin and Page, The anatomy of a large-scale hypertextual web search engine, 1998


Bonato, A course on the web graph, AMS 2008

Bryan and Leise, The $25,000,000,000 eigenvector, SIAM Review 2006
The job of a search engine is to receive queries and return a usable list of relevant matches, within in a reasonable time.
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What is the web?

The web is a distributed, linked collection of documents. This isn’t as obvious as it sounds:

- HTML or other content types?
- Static or dynamic?
- HTTP(S) or other protocols?
- Public or restricted?
It’s hard to tell how big, because estimates vary wildly and are constantly changing.

What counts as a web page: a URL, or the content returned? The “surface web” or the “deep web”?

Google (2008) claimed to have identified 1 trillion URLs, but they only index a fraction of those.

The size of the “indexed web” today is probably measured in the 10’s of billions.
A Google query on *a* finds over 25 billion results.

A breadth-first search rooted at http://www.math.unl.edu found 21,000 internal pages. What percentage of UNL is the Math Dept? What percentage of the web is UNL? Surely

$$20,000 \times 50 \times 10,000 = 10^{10}$$

is a huge underestimate.
How does a search engine work?
The need for ranking

A Google query on “cat” found 591,000,000 results. A search for “PageRank” got 81,000,000.

1. Word/term frequency
2. Word/term context (h1, h2, strong, etc.)
3. Back-link counts

All very vulnerable to SEO spamming.
PageRank – and other ranking algorithms, e.g., HITS – use global link analysis.
Let $W$ be the web-graph. Vertices are pages and there is a directed edge from $u$ to $v$ if a hyperlink, `<a href="...">cat</a>`, is found in $u$, pointing to $v$. (Ignore multiple links and loops.)

Let $n = |W| \ (n \sim 10^{10})$.

Seek a single vector $r \in \mathbb{R}^n$, with

1. $r_i \geq 0$
2. $\|r\|_1 = 1$

(i.e., stochastic), where each $r_i$ represents the relative importance of page $v_i$. 
PageRank: The goal

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\[ r_i = \sum_{v_j \rightarrow v_i} \frac{1}{d_j} \cdot r_j \]
What’s important?

Let $A$ be the adjacency matrix of the directed graph $W$ (i.e., $a_{i,j} = 1$ if $v_i \rightarrow v_j$, otherwise zero).

Let $D = diag(d_1^+, \ldots, d_n^+)$. Let $A_0 = D^{-1}A$ (allowing for non-invertibility)

Then

$$r = rA_0$$

In other words, find an eigenvector (the eigenvector?) of $A_0$ for $\lambda = 1$. 

Example

\[ A_0 = \begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 
\end{bmatrix} \]
Problems

Sinks

There are sure to be sinks in $W$.

If $W$ is a chain then

$$A_0 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots \\
\end{bmatrix}$$

which is nilpotent and so $sp(A_0) = \{0\}$

i.e., solutions to $rA_0 = r$ do not exist.
$W$ is not strongly connected or even connected.

$$A_0 = \begin{bmatrix} A' & * \\ 0 & A'' \end{bmatrix}$$

The multiplicity of $\lambda = 1$ is greater than 1.

I.e., solutions to $rA_0 = r$ are not unique.
Imagine a (finite state, discrete time, time-homogenous) Markov Process on $W$.

At each step the surfer clicks a link uniformly at random from the links on her current page.

If the page has no outlinks, pick a page uniformly at random from $W$. The transition probabilities for this process are

$$A_1 = A_0 + \frac{1}{n} z^T 1$$

where $z$ is the indicator vector for the sinks ($z_i = 1$ if $d_i^+ = 0$ and is 0 otherwise), and $1 = (1, 1, \ldots, 1)$. 
Example

\[
A_1 = \begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0
\end{bmatrix} + \frac{1}{4} \begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix} [1, 1, 1, 1] = \begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{bmatrix}
\]
Random surfer model

The transition matrix

\[ A_1 = A_0 + \frac{1}{n} z^T 1 \]

\[ = D^{-1} A + \frac{1}{n} z^T 1 \]

is a row-stochastic matrix.
Random surfer model

The stationary distribution of the process is the long-term proportion of the time that the surfer will spend on each page. If \( p = (p_i) \) is the stationary distribution then

\[
p = pA_1
\]

and so we are still seeking an eigenvector for \( \lambda = 1 \), but now of our modified matrix, \( A_1 \).
Lemma

If $S$ is a (row) stochastic matrix then $\lambda = 1$ is an eigenvalue.

Proof.

$S\mathbf{1}^T = \mathbf{1}^T$.

In other words, $\mathbf{1}^T$ is a right eigenvector, and so there must exist left eigenvectors too.
Perron’s Theorem

Theorem

Let $P > 0$ and let $\rho$ be the spectral radius of $P$. Then...

1. ...$\rho$ is positive and is an eigenvalue of $P$,
2. ...$\rho$ has left and right eigenvectors with positive entries,
3. ...$\rho$ has algebraic & geometric multiplicity 1, and
4. ...all the other eigenvalues are less than $\rho$ in magnitude.

Proof.

Find a fixed point of $Px/\|Px\|_1$ on $x_i \geq 0$, $\sum x_i = 1$...
So if $P$ is a positive row-stochastic matrix, and $x$ is a positive left eigenvector for $\rho$, then

$$\|x\|_1 = x1^T = x(P1^T) = (xP)1^T = \rho x1^T = \rho \|x\|_1$$

and so

$$\rho = 1$$
But there’s still a problem... 

Our transition matrix

\[ A_1 = D^{-1} A + \frac{1}{n} z^T 1 \]

isn’t positive.

(If \( A_1 \) were irreducible we could use the Perron-Frobenius Theorem.)

It’s the same issue as before; failure of (strong) connectedness.
Adapt the random surfer model

Imagine now at each step that the random surfer either

... clicks a link uniformly at random from the links on her current page

... or else ...

with probability $\alpha$ jumps to a new page chosen uniformly at random from $W$.

The probability $\alpha$ is called the teleportation constant.
Adapt the random surfer model

The new transition matrix is

\[ A_2 = (1 - \alpha)(D^{-1}A + \frac{1}{n}z^T1) + \alpha\frac{1}{n}1^T1 \]

This is often called the Google Matrix.

Clearly this is positive, stochastic.

Brin & Page (1998) report using \( \alpha = 0.15 \) in early Google.
### Example

The PageRank algorithm involves computing the importance of web pages based on their connectivity. The PageRank of a page is calculated by considering the PageRanks of other pages that link to it. Here is an example:

- **Graph:**
  - Page A points to page B.
  - Page B points to page D.
  - Page D points to page C.
  - Page C points to page A.

- **Transition Matrix (A):**
  
  \[
  A_2 = \begin{bmatrix}
  0.0375 & 0.3208 & 0.3208 & 0.3208 \\
  0.2500 & 0.2500 & 0.2500 & 0.2500 \\
  0.0375 & 0.4625 & 0.0375 & 0.4625 \\
  0.2500 & 0.2500 & 0.2500 & 0.2500 
  \end{bmatrix}
  \]

- **Page Rank (p):**
  
  \[
  p = \begin{bmatrix}
  0.1683 & 0.3078 & 0.2160 & 0.3078 
  \end{bmatrix}
  \]
We need to solve

\[ pA_2 = p \quad \text{or} \quad p(A_2 - I) = 0 \]

Gauss-Jordan elimination is \( O(n^3) \), or \( \sim 10^{30} \).

Moreover, it requires storage of the entire array, \( O(n^2) \), or \( \sim 10^{20} \) bytes (1 petabyte \( \sim 10^{12} \) bytes).
Let

\[ p_0 = \frac{1}{n} \mathbf{1} \]

\[ p_{k+1} = p_k A_2 \]

so that \( p_k = p_0 A_2^k \).

Since \( p_k \) is a product of row stochastic matrices, it is row stochastic.

Thus, if \( p_k \) converges, it converges to the normalized eigenvector (a.k.a., stationary distribution)
By Perron’s Theorem, $A_2$ is similar to a block Jordan matrix

$$
\begin{bmatrix}
1 \\
J_{\lambda_2}^{(m_2)} \\
& J_{\lambda_3}^{(m_3)} \\
& & \ddots
\end{bmatrix}
$$

where the eigenvalues of $A_2$ are

$$1 > \lambda_2 > \lambda_3 > \cdots > \lambda_N$$

each with multiplicity $m_i$. (In particular, $m_1 = 1$.)
The PageRank Algorithm

John Orr

Introduction

PageRank Computation

Further issues

Power method

But does it converge?

The powers of the Jordan blocks, $(J_{\lambda_i}^{(m_i)})^k$ converge to $0_{m_i \times m_i}$ and the rate of convergence is $O(\lambda_i^k)$.

Thus

1. $A_2^k$ converges to $1^T p$
2. $p_k$ converges to $p$, (independent of $p_0$, in fact) and
3. the rate of convergence is $O(\lambda_2^k)$. 
Power method
Complexity

\[ p_{k+1} = p_k A_2 \]

\[ = (1 - \alpha) p_k D^{-1} A + \frac{1}{n} p_k z^T 1 + \frac{\alpha}{n} p_k 1^T 1 \]

Most pages can be expected to contain a bounded number of outlinks. Empirical studies suggest the average number of outlinks per page is around 10. Thus \( A \) is sparse, and computing \( p_k D^{-1} A \) is also \( O(n) \).

Each iteration is \( O(n) \) operations. All operations are matrix-vector and from the form of the vectors (diagonal, rank-1, and sparse) storage is also \( O(n) \).

The following analysis casts light on the rapid convergence...
Theorem (Haveliwala & Kamavar, 2003)

If the eigenvalues of the stochastic matrix $A_1$ are

$$\{1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$$

then the eigenvalues of

$$A_2 = (1 - \alpha)A_1 + \frac{\alpha}{n}1^T1$$

are

$$\{1, (1 - \alpha)\lambda_2, (1 - \alpha)\lambda_3, \ldots, (1 - \alpha)\lambda_n\}$$

Corollary

The power method computation of the PageRank vector converges $O((1 - \alpha)^k)$. 
Power method
Rate of convergence

Proof (Langeville & Meyer, 2005)

Observe

\[ A_1 \mathbf{1}^T = \mathbf{1}^T \text{ and } \frac{1}{n} (\mathbf{1}^T \mathbf{1}) \mathbf{1}^T = \mathbf{1}^T \]

and so, wrt a basis that starts with \( \mathbf{1} \),

\[ A_2 = (1 - \alpha) A_1 + \frac{\alpha}{n} \mathbf{1} \mathbf{1}^T \]

\[ = (1 - \alpha) \begin{bmatrix} 1 & * \\ 0 & B \end{bmatrix} + \alpha \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & * \\ 0 & (1 - \alpha)B \end{bmatrix} \]
The web is constantly changing, and so rankings are not useful unless they are stable under small perturbations of $W$.

**Theorem (Ng, Zheng, Jordan 2001)**

Let $G$ be the PageRank matrix defined on a directed graph $W$ and let $p$ be its stationary distribution. Suppose $W'$ is obtained by changing the outlinks of vertices $i_1, i_2, \ldots, i_k$, and let $G'$ and $p'$ be the corresponding perturbations of $G$ and $p$. Then

$$
\|p' - p\|_1 \leq \frac{2 \sum_{j=1}^{k} p_{ij}}{\alpha}
$$
Variants of PageRank

“Intelligent surfer” transition matrix, $A'_1$ with values computed from server logs.

“Personalized teleportation vector”, $v$, gives

$$(1 - \alpha)A'_1 + \frac{\alpha}{n}1^Tv$$

The complexity of the calculation makes genuinely personalized vectors impractical.