Classifying Stable Ideals of Nest Algebras

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Introduction

Lecture plan:

- Nest algebras and their ideals
- Stable ideals
- Examples
- Characterization
- Classification
- Applications
A nest, $\mathcal{N}$, is a complete, linearly ordered lattice of projections.

$$\text{Alg} \mathcal{N} = \{ X : N^\perp X N = 0 \}$$
A nest, \( \mathcal{N} \), is a complete, linearly ordered lattice of projections.

\[ \text{Alg} \mathcal{N} = \{ X : N^\perp X N = 0 \} \]

Mostly, we use \textit{continuous} nests.
There is a very rich selection of norm-closed ideals.

- Weakly closed ideals
- Radicals
- Compact and compact-like
Weakly Closed Ideals

Theorem 1 (Erdos-Power, ’82). \( \mathcal{I} \) is a weakly closed ideal of \( \text{Alg}\,\mathcal{N} \) if and only if there is a increasing map \( \theta : \mathcal{N} \to \mathcal{N} \) satisfying \( \theta(N) \leq N \) such that

\[
\mathcal{I} = \{ X \in \text{Alg}\,\mathcal{N} : \theta(N)\downarrow XN = 0 \ \forall N \in \mathcal{N}\}
\]
**Weakly Closed Ideals**

**Theorem 1 (Erdos-Power, ’82).** \( I \) is a weakly closed ideal of \( \text{Alg} \mathcal{N} \) if and only if there is a increasing map \( \theta : \mathcal{N} \to \mathcal{N} \) satisfying \( \theta(N) \leq N \) such that

\[
I = \{ X \in \text{Alg} \mathcal{N} : \theta(N) \perp X N = 0 \quad \forall N \in \mathcal{N} \}.
\]
Definition 2. For $X \in \text{Alg}\mathcal{N}$, define

$$i^+_N(X) := \inf_{M > N} \| (M - N)X(M - N) \|$$

$$i^-_N(X) := \inf_{M < N} \| (N - M)X(N - M) \|$$
The Radical

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**The Radical**

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\]

**Theorem 2 (Ringrose, ’65).** The Jacobson Radical, \( \mathcal{R}_\mathcal{N} \), of \( \text{Alg}\, \mathcal{N} \) is equal to

\[
\{ X \in \text{Alg}\, \mathcal{N} : i^+_N(X) = i^-_N(X) = 0 \quad \forall N \in \mathcal{N} \}
\]
The Strong Radical

Let $\mathcal{N}$ be a continuous nest.

**Theorem 3 (O., '94).** Used $i^+_N$ seminorms to classify the lattice of ideals generated by maximal two-sided ideals. Showed that the strong radical is

$$ \{ X \in \text{Alg} \mathcal{N} : i^+_N(X) = 0 \text{ on a nowhere dense set} \} $$
Let $\mathcal{N}$ be a continuous nest.

**Theorem 4 (O., ’94).** Used $i^n_N$ seminorms to classify the lattice of ideals generated by maximal two-sided ideals. Showed that the strong radical is

$$\{ X \in \text{Alg} \mathcal{N} : i^n_N(X) = 0 \text{ on a nowhere dense set} \}$$

**Remark 4.** The strong radical for $\text{Alg} \mathbb{Z}^+$ is unknown.
The compact operators, $\mathcal{K}$, of $\text{Alg}.N$ are an ideal
Compact & Compact Character

- The compact operators, $\mathcal{K}$, of $\text{Alg}\, \mathcal{N}$ are an ideal.
- Call $X \in \text{Alg}\, \mathcal{N}$ compact character if
  $$(M - N)X(M - N)$$
is compact for all $0 < N < M < I$ in $\mathcal{N}$. 
Compact Character

A *ideal* is of compact character if all its elements are.

Example:
Compact Character

A *ideal* is of compact character if all its elements are.

Example:

\[
\mathcal{K}^+ := \{ X \in \text{Alg} \mathcal{N} : N \perp X N \perp \in \mathcal{K} \quad \forall N > 0 \}\]
A *ideal* is of compact character if all its elements are.

Example:

\[
\mathcal{K}^- := \{ X \in \text{Alg} \mathcal{N} : \ R X N N \in \mathcal{K} \quad \forall N < I \}
\]
Compact Character

A *ideal* is of compact character if all its elements are. Example:

\[ \mathcal{K}^- \cap \mathcal{K}^+ \]
Definition 5. A closed two-sided ideal, \( I \), is *stable* if \( \alpha(I) \subseteq I \) for all automorphisms \( \alpha \).
**Stable Ideals**

*Definition 5.* A closed two-sided ideal, $\mathcal{I}$, is *stable* if $\alpha(\mathcal{I}) \subseteq \mathcal{I}$ for all automorphisms $\alpha$.

*From here on, all nests are continuous*
Stable Ideals

Definition 5. A closed two-sided ideal, \( I \), is stable if \( \alpha(I) \subseteq I \) for all automorphisms \( \alpha \).

Examples:

- The trivial ideals \( 0 \) and \( \text{Alg} \mathcal{N} \)
- The compact operators
- The set of operators of compact character
- The Jacobson radical
- The strong radical
- Many more...
**Stable Ideals**

*Definition 5.* A closed two-sided ideal, $I$, is *stable* if $\alpha(I) \subseteq I$ for all automorphisms $\alpha$.

Non-Examples:

- Weakly closed ideals
- Larson’s ideal, $\mathcal{R}_N^\infty$
The lattice of 11 stable ideals of compact character
Automorphisms

Theorem 6 (Ringrose, ’66). Every isomorphism $\text{Alg } \mathcal{N}_1 \to \text{Alg } \mathcal{N}_2$ is of the form $\text{Ad}_S$, where $S$ is an invertible operator.
Automorphisms

Theorem 8 (Ringrose, ’66). Every isomorphism $\text{Alg} \mathcal{N}_1 \to \text{Alg} \mathcal{N}_2$ is of the form $\text{Ad}_S$, where $S$ is an invertible operator.

Theorem 8 (Davidson, ’84). If $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ is an order-dimension isomorphism then there is an invertible operator $S$ such that $\text{range}(SN S^{-1}) = \text{range}(\theta(N))$ for all $N \in \mathcal{N}_1$.
Automorphisms

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**Corollary 8.** $\text{Out}(\text{Alg} \mathcal{N}) \leftrightarrow \text{Aut}([0, 1])$
Theorem 9 (O., ’01). The set $\mathcal{J} \subseteq \text{Alg} \mathcal{N}$ is a stable ideal if and only if:
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- It is one of the eleven stable ideals of compact character, or
Theorem 9 (O., ’01). The set $\mathcal{J} \subseteq \text{Alg} \mathcal{N}$ is a stable ideal if and only if:

- It is one of the eleven stable ideals of compact character, or
- something horrid...
Main results:

- Simple, unified description of the stable ideals
- Classify the stable ideals
- Algebraic properties, quotient norms
Stable Nets

Let $P_1, P_2$ be two families of intervals of $\mathcal{N}$. 
Stable Nets

Let $P_1, P_2$ be two families of intervals of $\mathcal{N}$.

Needn’t be pairwise orthogonal!
Let $P_1, P_2$ be two families of intervals of $\mathcal{N}$.

Needn’t even be countable!!
Let $P_1, P_2$ be two families of intervals of $N$.

Definition 11. Say that $P_1$ refines $P_2$ if whenever $E \in P_1$ there is an interval $F \in P_2$ such that $E \leq F$. 
Let $P_1, P_2$ be two families of intervals of $\mathcal{N}$.

*Definition 11.* Say that $P_1$ refines $P_2$ if whenever $E \in P_1$ there is an interval $F \in P_2$ such that $E \leq F$.

*Definition 11.* A set, $\Omega$, of families of intervals is a *net of intervals* if it is a directed set under this ordering. $\Omega$ is a *stable net* if

$$\theta(P) := \{\theta(E) : E \in P\} \in \Omega$$

for all $\theta \in \text{Aut}([0, 1])$. 

**Stable Nets**
Theorem 12 (O., preprint ’05). The (non-zero) set $\mathcal{I} \subseteq \text{Alg} \mathcal{N}$ is a stable ideal if and only if there is a stable net $\mathcal{\Omega}$ such that $\mathcal{I}$ is

$$\{X \in \text{Alg} \mathcal{N} : \lim_{P \in \mathcal{\Omega}} \sup_{E \in P} \|EXE\|_{\text{ess}} = 0\}$$
Theorem 12 (O., preprint ’05). The (non-zero) set $\mathcal{J} \subseteq \text{Alg} \mathcal{N}$ is a stable ideal if and only if there is a stable net $\Omega$ such that $\mathcal{J}$ is

$$\{ X \in \text{Alg} \mathcal{N} : \limsup_{P \in \Omega, E \in P} \| EXE \|_{\text{ess}} = 0 \}$$

But what does it mean?!
Example 13. Let $\Omega$ be just the one family, $P = \{0\}$. Then

$$\lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}} = 0$$

for all $X$. This gives the ideal $J = \text{Alg} \mathcal{N}$. 
Example 13. Let $\Omega$ be just the one family, $P = \{I\}$. Then

$$\lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}} = \|X\|_{\text{ess}} = 0 \iff X \in \mathcal{K}$$

This gives the ideal $\mathcal{J} = \mathcal{K}$. 
Example 13. Let $\Omega$ consist of all singletons $\{N\}$ with $N > 0$. Then

$$\lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}} = \lim_{N \downarrow 0} \|NXN\|_{\text{ess}} = i_0^+(X)$$

This gives the kernel of $i_0^+$. 
Example 13. Let \( \Omega \) consist of the single family \( \{N : N < I\} \). Then

\[
\lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}} = \sup_{N < I} \|NXN\|_{\text{ess}} = 0
\]

\[\iff\]

\[X \in \mathcal{K}^{-}\]
Example 13. Let $\Omega$ consist of all finite partitions of $\mathcal{N}$. Then

$$\lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}} = \lim_{\{E_i\}} \sum_{i=1}^{n} \|E_iXE_i\| = 0$$

$\iff$

$$X \in \mathcal{R}_{\mathcal{N}}$$
Classification

When do two stable nets give the same ideal?
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Recall \( P_1 \geq P_2 \) if \( \forall E \in P_1 \exists F \in P_2 \text{ s.t. } E \leq F \).
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Say $\Omega_1$ is cofinal in $\Omega_2$ if $\forall P_2 \in \Omega_2 \exists P_1 \in \Omega_1$ s.t. $P_1 \geq P_2$. 
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$$\lim_{P \in \Omega_1} \sup_{E \in P} \|EXE\|_{\text{ess}} \leq \lim_{P \in \Omega_2} \sup_{E \in P} \|EXE\|_{\text{ess}}$$

and so $I_1 \supseteq I_2$. 
Classification Theorem

Theorem 14. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be stable ideals associated with stable nets $\Omega_1$ and $\Omega_2$. Then $\mathcal{J}_1 \supseteq \mathcal{J}_2$ if and only if $\Omega_1$ is cofinal in $\Omega_2$. 
Theorem 15. Let $I_1$ and $I_2$ be stable ideals associated with stable nets $\Omega_1$ and $\Omega_2$. Then $I_1 \supseteq I_2$ if and only if $\Omega_1$ is cofinal in $\Omega_2$.

Corollary 15. $I_1 = I_2$ if and only if $I_1$ and $I_2$ are mutually cofinal.
Assume $I_1 \supseteq I_2$
Sketch of Proof

Assume $J_1 \supseteq J_2$ and fix $Q_0 \in \Omega_2$. 
Sketch of Proof

Assume $I_1 \supseteq I_2$ and fix $Q_0 \in \Omega_2$. Goal: Find $P \in \Omega_1$ that refines $Q_0$. 
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Match up the inner and outer covers...
Theorem 16. Let $I$ be given by $\Omega$ and $X \in \text{Alg} \, \mathcal{N}$. Then

$$\|X + I\| = \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}}$$
**Quotient Norm**

**Theorem 16.** Let $J$ be given by $\Omega$ and $X \in \text{Alg}\mathcal{N}$. Then

$$\|X + J\| = \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}}$$

$$P_{T,a} := \{E : \|ETE < a\|_{\text{ess}}\} \quad T \in J, a > 0$$

$$\Omega' := \{P_{T,a} : T \in J, a > 0\}$$
Theorem 16. Let $\mathcal{I}$ be given by $\Omega$ and $X \in \text{Alg}\, \mathcal{N}$. Then

$$\|X + \mathcal{I}\| = \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}}$$

$$P_{T,a} := \{E : \|ETE < a\|_{\text{ess}}\} \quad T \in \mathcal{I}, a > 0$$

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Thus $\Omega'$ specifies $\mathcal{I}$
Theorem 16. Let \( \mathcal{I} \) be given by \( \Omega \) and \( X \in \text{Alg}\, \mathcal{N} \). Then

\[
\|X + \mathcal{I}\| = \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}}
\]

\[
P_{T,a} := \{E : \|ETE < a\|_{\text{ess}}\} \quad T \in \mathcal{I}, a > 0
\]

\[
\Omega' := \{P_{T,a} : T \in \mathcal{I}, a > 0\}
\]

Thus \( \Omega' \) specifies \( \mathcal{I} \)

\( \Rightarrow \) \( \Omega' \) and \( \Omega \) are mutually cofinal
**Quotient Norm**

**Theorem 16.** Let $I$ be given by $\Omega$ and $X \in \text{Alg} \mathcal{N}$. Then

$$\|X + I\| = \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}}$$

$$P_{T,a} := \{E : \|ETE < a\|_{\text{ess}}\} \quad T \in I, a > 0$$

$$\Omega' := \{P_{T,a} : T \in I, a > 0\}$$

Thus $\Omega'$ specifies $I$

$$\implies \Omega' \text{ and } \Omega \text{ are mutually cofinal}$$

$$\implies \lim_{P \in \Omega'} \sup_{E \in P} \|EXE\|_{\text{ess}} = \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}}$$
Theorem 17. $J_1, J_2$ stable ideals $\implies J_1 + J_2$ stable ideals.
Theorem 17. \( J_1, J_2 \) stable ideals \( \Longrightarrow \) \( J_1 + J_2 \) stable ideals.

How is net for \( J_1 + J_2 \) related to \( J_1, J_2 \)?
Algebra of Ideals

Theorem 17. \( J_1, J_2 \) stable ideals \( \implies J_1 + J_2 \) stable ideals.

Let \( \Omega_1, \Omega_2 \) be stable nets. For \( P_1 \in \Omega_1 \) and \( P_2 \in \Omega_2 \) define

\[ P_1 \cdot P_2 := \{ E_1 E_2 : E_1 \in P_1, E_2 \in P_2 \} \]

and then define

\[ \Omega_1 \cdot \Omega_2 := \{ P_1 \cdot P_2 : P_1 \in \Omega_1, P_2 \in \Omega_2 \} \]
Theorem 17. \( I_1, I_2 \) stable ideals \( \implies I_1 + I_2 \) stable ideals.

Theorem 17. \( \Omega := \Omega_1 \cdot \Omega_2 \) is a stable net, and

\[
I_1 + I_2 = \{ X \in \text{Alg} \mathcal{N} : \lim_{P \in \Omega} \sup_{E \in P} \| E X E \|_{\text{ess}} = 0 \}
\]