(1) We will do this in three cases, showing that \( \lim_{x \to a} |x| = |a| \) for \( a = 0 \), \( a > 0 \) and \( a < 0 \). In each case, let \( \epsilon > 0 \) be given and we will find an appropriate \( \delta > 0 \) so that if \( 0 < |x - a| < \delta \), then \( ||x| - |a|| < \epsilon \).

If \( a = 0 \), then letting \( \delta = \epsilon \), we see that if \( 0 < |x - 0| = |x| < \delta \), then \( ||x| - |0|| = ||x|| = |x| < \delta = \epsilon \) and we are done.

If \( a > 0 \), let \( \delta = \min\{\epsilon, a\} \). Then, for any \( x \) such that \( 0 < |x - a| < \delta \), \( x > 0 \), and so \( |x| = x \). Thus, if \( 0 < |x-a| < \delta \), \( ||x| - |a|| = |x-a| < \epsilon \), since \( |a| = a \).

If \( a < 0 \), let \( \delta = \min\{\epsilon, |a|\} \). Then, for any \( x \) such that \( 0 < |x-a| < \delta \), \( x < 0 \), so \( |x| = -x \). Thus, if \( 0 < |x-a| < \delta \), \( ||x| - |a|| = |a-x| = |x-a| < \epsilon \).

(b) Since \( f \) is continuous on \((a,b)\) and \( \text{abs}(x) = |x| \) is continuous everywhere by problem 1(a), we know that by Theorem 2.5.3, which says that if two functions are continuous at a point, then the composition of those two functions is continuous at that point, that \( \text{(abs} \circ f)(x) = |f(x)| = g(x) \) is continuous at every point of \((a,b)\) and thus is continuous on \((a,b)\).

(2) We proved in class that \( \lim_{x \to 0} x \sin(1/x) = 0 \). Thus, since \( f(0) = 0 \), we know that \( f \) is continuous at 0. If \( a \neq 0 \), we must show that \( f(x) \) is continuous at \( a \). However, we know that \( 1/x \) is continuous for all \( x \neq 0 \) since it is a rational function. We are given that \( \sin(x) \) is continuous for all \( x \in \mathbb{R} \), and thus by Theorem 2.5.3, we know that \( \sin(1/x) \) is continuous for all \( x \neq 0 \). \( x \) is a polynomial and is thus continuous everywhere, and we know that the product of two continuous functions is continuous, so \( x \sin(1/x) \) is continuous at all \( a \) such that \( a \neq 0 \). Thus, \( f(x) \) is continuous everywhere.

(3) Note that if \( a = 0 \), then \( 0^n = 0 \) and so the claim holds. Let \( p(x) = x^n \).

On the other hand, if \( a > 0 \), we simply want to find \( b \) and \( c \) such that \( p(b) < a < p(c) \). If we do this, we are done, since \( p(x) \) is a polynomial and thus is continuous everywhere. In particular, \( p(x) \) is continuous on \([b,c]\). Thus, by the Intermediate Value Theorem, we know that there exists an \( x \in [a,b] \) such that \( p(x) = a \). That is, \( x^n = a \). Note that if we let \( b = 0 \), then \( p(0) = 0 \) and \( a > 0 \). On the other hand, let \( c = 2a \). Then \( p(2a) = (2a)^n = 2^n a^n \geq 2a > a \). Thus, \( p(0) < a < p(2a) \) and we are done by the above argument.

(4) \( \text{abs}(x) = |x| \) is absolutely continuous on \( \mathbb{R} \). Before we prove this, we would like to show that \( ||x| - |y|| \leq |x - y| \) for all \( x, y \in \mathbb{R} \). But \( ||x| - |y|| = ||x - y + y - y|| \leq ||x - y| + |y| - |y|| = ||x - y|| = |x - y| \), where the inequality above is due to the triangle inequality. Let \( \epsilon > 0 \) be given. Then let \( \delta = \epsilon \). If \( |x - y| < \delta \), then \( ||x| - |y|| \leq |x - y| < \delta = \epsilon \). Thus, \( \text{abs}(x) \) is absolutely continuous on \( \mathbb{R} \).
(5) (a) Let $\epsilon > 0$ be any positive real number. Let $\delta > 0$ be such that if $|x - y| < 2\delta$ and $x, y \in (a, b)$, then $|f(x) - f(y)| < \epsilon$. Consider the open cover of $(a, b)$ made up of intervals of the form $O_n = (a + n\delta, a + (n + 2)\delta)$ for $n \geq 0$ an integer. Is this cover finite? By the Archimedean property applied to $\delta$ and $b - a$, there exists an $N' \in \mathbb{N}$ such that $N'\delta > b - a$. But this gives that $a + N'\delta > b$. Let $N$ be the largest integer such that $a + (N + 2)\delta < b$. Then, if we take \{O_n : 0 \leq n \leq N - 1, n \in \mathbb{Z}\} \cup (a + (N + 1)\delta, b)$, we have a finite open cover of $(a, b)$. Let $O_N = (a + (N + 1)\delta, b)$. For each $n \in \mathbb{Z}$ with $0 \leq n \leq N$, let $x_n \in O_n$. Let $M' = \max\{f(x_0), f(x_1), \ldots, f(x_N)\}$. Then, for any $x \in (a, b)$, there is an $n$ such that $x \in O_n$. Since $x \in O_n$ and $x_n \in O_n$, $|x - x_n| < 2\delta$. Thus, by our choice of $\delta$, we know that $|f(x) - f(x_n)| < \epsilon$. But this implies that $f(x) - f(x_n) < \epsilon$, and so $f(x) < f(x_n) + \epsilon \leq M' + \epsilon$. Let $M = M' + \epsilon$ and since $x$ was any number in $(a, b)$, we have seen that for all $x \in (a, b)$, $f(x) \leq M$.

(b) Consider $f(x) = x$ on $(0, 1)$. This function never attains a maximum or a minimum on $(0, 1)$. 
