1. In this question we will develop from scratch the natural logarithm function and some of its properties. We define the function \( \ln x \) for \( x > 0 \) as follows. Let \( f(t) = 1/t \) and set

\[
\ln x = \int_1^x f.
\]

(a) Show that \( \ln x \) is defined for all \( x > 0 \), i.e., that \( f \) is integrable over the relevant intervals.

(b) Show that \( \ln x \) is continuous everywhere on its domain.

(c) Show that \( d/dx[\ln x] = 1/x \).

(d) Show that \( \ln x \) is a strictly increasing function.

(e) Show that \( \ln x \) is invertible.

(f) Show, by exhibiting suitable partitions, that

\[
\ln 2 < 1 < \ln 3.
\]

(g) Show that there is a unique solution in \((2, 3)\) to \( \ln x = 1 \).

(h) Prove that for \( a, b > 0 \) we have

\[
\ln(ab) = \ln a + \ln b.
\]

(Hint: Define a function \( L(x) = \ln(ax) \) and prove that \( L'(x) = 1/x \). Deduce that \( L(x) = \ln(x) + c \) for some constant \( c \) and deduce the value of \( c \))

(i) Prove that \( \ln(x^n) = n \ln x \) for all \( x > 0 \) and non-negative integers \( n \).

(j) Prove that \( \ln(a/b) = \ln a - \ln b \) for all \( a, b > 0 \).

(k) Prove that \( \ln(x^n) = n \ln x \) for all integers \( n \).

2. Let \( u \) be a differentiable function on \([a, b]\) and suppose that \( u' \) is integrable on \([a, b]\). Let \( f \) be a function that is continuous on the range of \( u \). If \( u(a) = c \) and \( u(b) = d \) prove that

\[
\int_a^b f(u(x))u'(x)dx = \int_c^d f(x)dx.
\]

(Hint: As in Theorem 3.6.1, let \( F(x) = \int_c^x f(t)dt \) for all \( x \) in the range of \( u \). Let \( g(x) = F(u(x)) \) for \( x \in [a, b] \). Now compute \( \int_a^b g' \) in two different ways.)