(1) A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy-duty fencing selling for \$3 dollars a foot, while the remaining two sides will use standard fencing selling for \$2 a foot. The fence is to cost \$6000. How much of each type of fencing should be bought to fence the rectangular plot of greatest area for that cost?

Show all of your work, clearly define your variables, explain your equations, and be sure to show that your answer does give the maximum area.

Let \( x \) represent amount of \$3 per foot fencing, and let \( y \) be the amount of \$2 per foot fencing. Thus the total cost of the fence is \( 3x + 2y \), and since the fence should cost \$6000, we have \( 3x + 2y = 6000 \). Since \( x \) and \( y \) each represent the amount of fencing used for two sides of the rectangular plot, the area of the plot, \( A \), is given by the equation

\[
A = \frac{xy}{2},
\]

In order to maximize \( A \), we must eliminate one of the variables. We solve the equation \( 3x + 2y = 6000 \) for \( y \) to get \( y = \frac{3}{2}(6000 - 3x) = 3000 - \frac{3}{2}x \). Therefore

\[
A = \frac{1}{4}x(3000 - \frac{3}{2}x) = 750x - \frac{3}{8}x^2.
\]

Since \( A \) must be positive, it follows that the inequalities \( x \geq 0 \) and \( 3000 - \frac{3}{2}x \geq 0 \) must hold. The second inequality implies that \( x \leq 2000 \). Thus, we are looking for the maximum value of \( A \) on the closed bounded interval \([0, 2000]\). We now take the derivative of \( A \) with respect to \( x \) in order to find any critical points in the region. We calculate \( A' = 750 - \frac{3}{4}x \) and see that \( A' = 0 \) when \( 750 - \frac{3}{4}x = 0 \), or when \( x = 1000 \). Since \( A' \) is always defined, \( x = 1000 \) is the only critical point. Thus the maximum area will occur at either \( x = 0, 1000 \) or \( 2000 \). (Since the interval we are looking at is closed and bounded, we know that \( A \) will have an absolute maximum on this interval, and it must occur at a critical point or at an endpoint.) Hence, we evaluate \( A \) at each of these points:

\[
\begin{align*}
  x = 0 & : \\
  A = (750)(0) - \frac{3}{8}(0)^2 = 0 \\

  x = 1000 & : \\
  A = 750(1000) - \frac{3}{8}(1000)^2 = 375000 \\

  x = 2000 & : \\
  A = (750)(2000) - \frac{3}{8}(2000)^2 = 0
\end{align*}
\]

Of these three points, \( A \) takes on the highest value at \( x = 3 \). Therefore, the maximum area of the rectangular plot occurs when \( x = 1000 \). When \( x = 1000 \), the value of \( y \) is
\[ y = 3000 - \frac{3}{2}(1000) = 3000 - 1500 = 1500. \] Hence, 1000 feet of the $3 fencing and 1500 feet of the $2 fencing should be bought to fence in the greatest area for $6000. The plot will have an area of 375,000 square feet.

(2) (a) The graph of the function \( f(x) = \cos x - x \) on the interval \([-3, 3]\) is graphed below. The point \( x_0 = -0.5 \) is an initial guess for the root of \( f \). Using the graph to sketch key pieces, explain how the first iteration of Newton’s method is found. You do not necessarily need to give the formula for Newton’s method, but you should clearly explain what point is found.

Given \( x_0 = -0.5 \), the next iterate of Newton’s method is found by looking at the line tangent to \( f(x) \) at \(-0.5\) and then determining where this tangent line intersects the \( x\)-axis. The line drawn in the figure above is the tangent line at the point \((-0.5, \cos(-0.5) + 0.5) \approx (-0.5, 1.378)\) which has equation \( y = (-\sin(-0.5) - 1)(x + 0.5) - (\cos(-0.5) - 0.5) \approx -0.521x + 1.117 \). The point \( x_1 \) is the \( x\)-intercept of this line and is the value given after one iteration of Newton’s method.

(b) Using \( x_0 = -0.5 \) as an initial guess for the root of \( f(x) = \cos x - x \), find \( x_2 \), the second iterate of Newton’s method. Show all of your work.

The formula for Newton’s method is

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \]

Since \( f(x) = \cos x - x \), its derivative is \( f'(x) = -\sin x - 1 \). Thus the formula for Newton’s method in this case is

\[ x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1}. \]

Substituting \( x_0 = -0.5 \) into the above formula shows that

\[ x_1 = -0.5 - \frac{\cos(-0.5) - 0.5}{-\sin(-0.5) - 1} \approx 2.1462738. \]

Therefore,

\[ x_2 = x_1 - \frac{\cos x_1 - x_1}{-\sin x_1 - 1} \approx 2.1462738 - \frac{\cos(2.1462738) - 2.1462738}{-\sin(2.1462738) - 1} \approx 0.68319173. \]
(c) **Approximate the root of** $f$ **to 5 decimal places.**

We repeat the above procedure until the first 5 decimal places of the iterations are the same for several iterations. The following values were obtained:

$$x_3 \approx .7398163 \quad x_4 \approx .7390853 \quad x_5 \approx .7390851 \quad x_6 \approx .7390851 \quad x_7 \approx .7390851$$

Therefore the root of $f$ is approximately .73908 (.73909 if you round up). (Note: it was not necessary to do 7 iterations. We could be fairly confident of the value after 5 iterations.)

(3) **Consider the function** $f(x) = x^5 - 5x^3 + 2$. **For the following, show all of your work and justify your answers to receive full credit.**

(a) **Specify the intervals on which** $f$ **is increasing and those on which** $f$ **is decreasing.**

The first derivative gives information about where $f$ is increasing or decreasing. Thus, we must calculate $f'(x)$ and find any critical points. The derivative of $f$ is

$$f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3),$$

which is defined everywhere and which is 0 at $x = 0, \pm \sqrt{3}$. Thus the only places where $f$ can go from increasing to decreasing or from decreasing to increasing are at these three points. Hence, we pick points in the intervals $(-\infty, -\sqrt{3}), (-\sqrt{3}, 0), (0, \sqrt{3})$ and $(\sqrt{3}, \infty)$ to determine the behavior of $f$. These values are summarized in the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>$-\sqrt{3}$</th>
<th>-1</th>
<th>0</th>
<th>$\sqrt{3}$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>20</td>
<td>0</td>
<td>-10</td>
<td>0</td>
<td>-10</td>
<td>0</td>
</tr>
</tbody>
</table>

Since $f$ is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$, we have that $f$ is increasing on $(-\infty, -\sqrt{3})$ and on $(\sqrt{3}, \infty)$ and decreasing on $(-\sqrt{3}, \sqrt{3})$.

(b) **Specify the intervals on which** $f$ **is concave up and those on which** $f$ **is concave down.**

The second derivative gives information about the concavity of $f$. The second derivative of $f$ is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

Since $f''(x)$ is defined everywhere, we will not have any inflection points occurring where the second derivative is not defined. We see that $f''(x) = 0$ when $x = 0, \pm \sqrt{\frac{3}{2}}$. Thus, we choose points from the intervals $(-\infty, -\sqrt{\frac{3}{2}}), (-\sqrt{\frac{3}{2}}, 0), (0, \sqrt{\frac{3}{2}})$ and $(\sqrt{\frac{3}{2}}, \infty)$; this is summarized in the following chart.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>$-\sqrt{\frac{3}{2}}$</th>
<th>-1</th>
<th>0</th>
<th>$\sqrt{\frac{3}{2}}$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f''(x)$</td>
<td>-100</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>-10</td>
<td>0</td>
</tr>
</tbody>
</table>
The function \( f \) is concave up when \( f''(x) > 0 \) and concave down when \( f''(x) < 0 \). Therefore, \( f \) is concave up on \((-\sqrt{\frac{3}{2}}, 0)\) and \((\sqrt{\frac{3}{2}}, \infty)\). It is concave down on \((-\infty, -\sqrt{\frac{3}{2}})\) and \((0, \sqrt{\frac{3}{2}})\).

(c) Find all local maxima and minima.

A local minimum occurs when the function changes from being decreasing to increasing. Therefore, \( f \) has a local minimum at \( x = \sqrt{3} \). The local minimum is \( f(\sqrt{3}) = -6\sqrt{3} + 2 \approx -8.392 \). A local maximum occurs when the function changes from being increasing to decreasing. Hence \( f \) has a local maximum at \( x = -\sqrt{3} \), and the local maximum is \( f(-\sqrt{3}) = 6\sqrt{3} + 3 \approx 12.392 \). Since \( f \) is decreasing to the left and to the right of \( x = 0 \), no local minimum or maximum occurs there — even though \( x = 0 \) is a critical point.

(d) Find all inflection points.

An inflection point occurs when the concavity of \( f \) changes. Therefore, \(-\sqrt{\frac{3}{2}}, 0\) and \(\sqrt{\frac{3}{2}}\) are all inflection points.

(e) Determine whether or not \( f \) has an absolute maximum or an absolute minimum. If so, identify the absolute maximum and minimum values and state where they occur. If not, explain why.

The function does not have an absolute minimum because as \( x \) approaches \(-\infty \), \( f(x) \) approaches \(-\infty \). Similarly, \( f \) does not have an absolute maximum because as \( x \) approaches \(\infty \), \( f(x) \) also approaches \(\infty \).