Instructions: Do any three problems.

(1) Let $k$ be a field and let $\emptyset \neq S \subseteq \mathbb{A}^m(k)$. Recall that $\text{affspan}(S) = \text{V}(\text{Lin}(S))$, where $\text{Lin}(S) = \{a_1x_1 + \cdots + a_mx_m + a_0 \mid k[x_1, \ldots, x_m] = k[A] ; S \subseteq \text{V}(a_1x_1 + \cdots + a_mx_m + a_0)\}$. For $p \in \text{affspan}(S)$, let $W_p(S) = \{v - p : v \in \text{affspan}(S)\}$. Show that $W_p(S)$ is a vector subspace of $k^m = \mathbb{A}^m(k)$ and that this subspace is independent of $p$ (i.e., $W_p(S) = W_q(S)$ for all $p \in \text{affspan}(S)$).

(2) Let $k$ be a field and let $\emptyset \neq S \subseteq \mathbb{A}^m(k)$. We define $\text{dim}\text{affspan}(S)$ to be the vector space dimension of $W_p(S)$ for any $p \in \text{affspan}(S)$ (this is well-defined by Problem (1)). Let $d = \text{dim}\text{affspan}(S)$. Show that there exist $d + 1$ points $p_0, \ldots, p_d \in S$ such that $\text{affspan}(p_0, \ldots, p_d) = \text{affspan}(S)$.

(3) Let $k$ be a field and let $\phi = (\phi_1, \ldots, \phi_m) : \mathbb{A}^1(k) \to \mathbb{A}^m(k)$ be a morphism. If $\text{dim}\text{affspan}(\text{im}(\phi)) \leq d$ for all $i$, show that dim $\text{affspan}(\text{im}(\phi)) \leq d$.

(4) Let $k$ be a field. Let $S \subseteq \mathbb{A}^m(k)$ and let $V$ be the Zariski closure of $S$. Show that $\text{affspan}(S) = \text{affspan}(V)$.

(5) Let $\phi = (\phi_1, \ldots, \phi_m) : \mathbb{A}^1(k) \to \mathbb{A}^m(k)$ be a morphism such that $m > 1$ and $\text{deg} \phi_i \leq 2$ for all $i$, not all constant. Let $C$ be the closure of $\phi(\mathbb{A}^1(k))$. Show that $\text{dim}\text{affspan}(C) \leq 2$. Conclude that $\text{Sec}_2(C)$ is contained in a 2-dimensional plane. [Aside: This shows that $\text{dim}\text{Sec}_2(C) \leq 2$, and thus that rational curves of degree at most 2 in $\mathbb{A}^m(k)$ with $m > 2$ are always defective. One can also show that any curve of degree at most 2 is rational, hence curves of degree at most 2 are always defective.]

(6) Let $L_1 \subseteq \mathbb{A}^3(k)$ be the line $V(x, y + 1)$ and let $L_2 \subseteq \mathbb{A}^3(k)$ be the line $V(y, x - 1)$, where $k[\mathbb{A}^3] = k[x, y, z]$. Let $P_1$ be the plane $V(y + 1)$ and let $P_2$ be the plane $V(y - 1)$. Let $L = L_1 \cup L_2$. Let $\sigma_2 : V^2 \times \Delta_2 \to \mathbb{A}^3(k)$, so $\text{Sec}_2(V)$ is the closure of the image of $\sigma_2$. Show that $\text{im}(\sigma_2) = (\mathbb{A}^3(k) \setminus (P_1 \cup P_2)) \cup (L_1 \cup L_2)$. [Hint: If $q \in \mathbb{A}^3(k) \setminus (P_1 \cup P_2)$, consider the intersections of the planes $Q_1$ and $Q_2$ where $Q_i$ contains $q$ and $L_i$.]

(7) Let $k$ be a field of characteristic not equal to 2 and let $V$ be the image of $\nu : \mathbb{A}^2(k) \to \mathbb{A}^5(k)$ defined by $\nu((a, b)) = (a^2, ab, b^2, a, b)$ (so the component functions of $\nu$ are the nontrivial monomials of degree at most 2); $V$ is known as the Veronese variety.

(a) Show that $V$ is closed and that $\nu$ is an isomorphism to its image.
(b) Show that the set of points $(p', q') \in V \times V$ such that $p' \neq q'$ and such that the line $\ell_{pq} \subseteq \mathbb{A}^2(k)$ through $p$ and $q$ is neither vertical nor horizontal and does not go through the origin, where $\nu(p) = p'$ and $\nu(q) = q'$, is a nonempty open subset $U' \subseteq V \times V$.
(c) Recall that the secant variety $Sec_2(V)$ is the closure of the image of $\sigma_2 : V^2 \times \Delta_2 \to \mathbb{A}^5(k)$, and hence the closure of the image of $f : A^2(k) \to A^5(k)$ defined as $f(a, b, c, d, e) = (e(a^2, ab, b^2, a, b) + (1 - c)(e^2, cd, d^2, c, d))$. Show in fact that $Sec_2(V)$ is contained in the closure of the image of $f : A^2(k) \to A^5(k)$ defined by $g(a, b, c, d) = c!((a, 0)) + d!((0, b)) + (1 - c - d)!((a/2, b/2))$. [Hint: For points $p \neq q \in A^2(k)$, let $p' = \nu(p)$ and $q' = \nu(q)$.
Show that the secant line $L_{pq} \subseteq A^2(k)$ through $p'$ and $q'$ lies in the affine span of the image $\nu(\ell_{pq})$ of the line $\ell_{pq} \subseteq A^2(k)$ through $p$ and $q$. Now use Problem (5).] (Aside: This implies that dim $Sec_2(V) \leq 4$ and hence that $Sec_2(V)$ is defective. In fact, dim $Sec_2(V) = 4$, so $Sec_2(V)$ is defined by a single polynomial equation on $A^5(k)$.]

(8) Let $k$ be a field of characteristic not 2 or 3, and let $C$ be the image of $\tau : \mathbb{A}^1(k) \to \mathbb{A}^3(k)$, defined as $\tau(t) = (t^2, t^3)$.

For any $a \in k$, the line $L_a$ through $p = \tau(a)$ with direction vector $(1, 2a, 3a^2)$ is called the tangent line to $C$ at $p$. Let $k[A] = k[t]$ and $k[A^3] = k[x, y, z]$. Let $f \in k[x, y, z]$ be a nonzero polynomial of degree 1.

(a) Show that $C$ is closed and that $\tau$ is an isomorphism to its image. (In fact, the image of $\mathbb{A}^1(k)$ under a morphism is always closed, this is harder to show.)
(b) Show that $\tau^*(f)$ has at most 3 roots, counted with multiplicity.
(c) Let $\tau(a) = p \in C$. If $L_p \subseteq V(f)$, show that $t = a$ is a root of multiplicity at least 2 (i.e., show that $(t - a)^2$ divides $\tau^*(f)$). [Aside: If $t = a$ is a root of multiplicity 3, we say $V(f)$ is an osculating plane for $C$ at $p$.]
(d) Show that if $q$ is on a tangent line but not on $C$, then $q$ is on no secant line of $C$. Conclude for any $q \neq p \in C$, if $q \in L_p$, then $q$ is not in the image of $\sigma_2 : C^2 \times \Delta_2 \to \mathbb{A}^5(k)$.}