Proof of Perron-Frobenius Theorem

Let \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) with \( y^T \) denoting exclusively the transpose of vector \( y \). Let \( \|x\| = \max_i \{|x_i|\} \) be the norm. Then the induced operator norm for matrix \( A = [a_{ij}] \) is \( \|A\| = \max_i \{|\sum_j a_{ij}|\} \).

Consider a Markov’s chain on \( n \) states with transition probabilities \( p_{ij} = \Pr(X_{k+1} = i|X_k = j) \), independent of \( k \), and \( P = [p_{ij}] \) the transition matrix. Then \( \sum_{i=1}^n p_{ij} = 1 \) for all \( j \). Let \( p_{ij}^{(t)} = \Pr(X_{k+t} = i|X_k = j) \) and \( P^{(t)} = [p_{ij}^{(t)}] \) be the \( t \)-step transition probability matrix. Then we have \( p_{ij}^{(t)} = \sum_k p_{ik}^{(t-1)} p_{kj} \) for all \( i, j \). In matrix, \( P^{(t)} = P^{(t-1)} P = \cdots = P^t \) which is the \( t \)-step transition matrix. If \( q = (q_1, \ldots, q_n)^T \) is a probability distribution for the Markovian states at a given iterate with \( q_i \geq 0, \sum q_i = 1 \), then \( PQ \) is again a probability distribution for the states at the next iterate. A probability distribution \( w \) is said to be a steady state distribution if it is invariant under the transition, i.e. \( Pw = w \). Such a distribution must be an eigenvector of \( P \) and \( \lambda = 1 \) must be the corresponding eigenvalue.

The existence as well as the uniqueness of the steady state distribution is guaranteed for a class of Markovian chains by the following theorem due to Perron and Frobenius.

**Theorem 1.** Let \( P = [p_{ij}] \) be a probability transition matrix, i.e. \( p_{ij} \geq 0 \) and \( \sum_{i=1}^n p_{ij} = 1 \) for every \( j = 1, 2, \ldots, n \). Assume \( P \) is irreducible and transitive in the sense that \( p_{ij} > 0 \) for all \( i, j \). Then \( 1 \) is a simple eigenvalue of \( P \) and all other eigenvalues \( \lambda \) satisfy \( |\lambda| < 1 \). Moreover, the unique eigenvector can be chosen to be a probability vector \( w \) and it satisfies \( \lim_{t \to \infty} P^t = [w, w, \ldots, w] \). Furthermore, for any probability vector \( q \) we have \( P^t q \to w \) as \( t \to \infty \).

**Proof.** We first prove a claim that \( \lim_{t \to \infty} p_{ij}^{(t)} \) exist for all \( i, j \) and the limit is independent of \( j \), \( \lim_{t \to \infty} p_{ij}^{(t)} = w_i \).

Because \( P = [p_{ij}] \) (is irreducible and transitive) has non-zero entries, we have

\[
\delta = \min_{ij} p_{ij} > 0.
\]

Consider the equation of the \( ij \)th entry of \( P^{t+1} = [p_{ij}^{(t+1)}] = P^t P \),

\[
p_{ij}^{(t+1)} = \sum_k p_{ik}^{(t)} p_{kj}.
\]

Let \( 0 < m_i^{(t)} := \min_j p_{ij}^{(t)} \leq \max_j p_{ij}^{(t)} := M_i^{(t)} < 1 \).
Then, we have

\[ m_i^{(t+1)} = \min_j \sum_k p_{ik}^t p_{kj} \geq m_i^{(t)} \sum_k p_{kj} = m_i^{(t)}. \]

i.e., the sequence \( \{m_i^{(1)}, m_i^{(2)}, \ldots\} \) is non-decreasing. Similarly, the upper bound sequence \( \{M_i^{(1)}, M_i^{(2)}, \ldots\} \) is non-increasing. As a result, both limits \( \lim_{t \to \infty} m_i^{(t)} = m_i \leq M_i = \lim_{t \to \infty} M_i^{(t)} \) exist. We now prove they are equal \( m_i = M_i \).

To this end, we consider the difference \( M_i^{(t+1)} - m_i^{(t+1)} \):

\[
M_i^{(t+1)} - m_i^{(t+1)} = \max_j \sum_k p_{ik}^t p_{kj} - \min_{\ell} \sum_k p_{ik}^t p_{k\ell} \\
= \max_j \sum_k p_{ik}^t (p_{kj} - p_{k\ell}) \\
= \max_j \left[ \sum_k p_{ik}^t (p_{kj} - p_{k\ell})^+ + \sum_k p_{ik}^t (p_{kj} - p_{k\ell})^- \right] \\
\leq \max_{j,\ell} \left[ M_i^{(t)} \sum_k (p_{kj} - p_{k\ell})^+ + m_i^{(t)} \sum_k (p_{kj} - p_{k\ell})^- \right].
\]

(1)

where \( \sum_k p_{ik}^t (p_{kj} - p_{k\ell})^+ \) means the summation of only the positive terms \( p_{kj} - p_{k\ell} > 0 \) and similarly \( \sum_k p_{ik}^t (p_{kj} - p_{k\ell})^- \) means the summation of only the negative terms \( p_{kj} - p_{k\ell} < 0 \).

It is critical to notice the following unexpected equality with the notations \( \sum_k^- (p_{kj} - p_{k\ell}) := \sum_k (p_{kj} - p_{k\ell})^- \), \( \sum_k^+ (p_{kj} - p_{k\ell}) := \sum_k (p_{kj} - p_{k\ell})^+ \):

\[
\sum_k^- (p_{kj} - p_{k\ell}) = \sum_k^- (p_{kj} - p_{k\ell}) \\
= \sum_k^- p_{kj} - \sum_k^- p_{k\ell} \\
= 1 - \sum_k^- p_{kj} - (1 - \sum_k^+ p_{k\ell}) \\
= \sum_k^+ (p_{k\ell} - p_{kj}) \\
= - \sum_k^- (p_{kj} - p_{k\ell})^+.
\]

Hence, the inequality (1) becomes

\[
M_i^{(t+1)} - m_i^{(t+1)} \leq (M_i^{(t)} - m_i^{(t)}) \max_{j,\ell} \sum_k (p_{kj} - p_{k\ell})^+.
\]

If \( \max_{j,\ell} \sum_k (p_{kj} - p_{k\ell})^+ = 0 \), it is done that \( M_i^{(t)} = m_i^{(t)} \). Otherwise, for the pair \( j, \ell \) that gives the maximum let \( r \) be the number of terms in \( k \) for which \( p_{kj} - p_{k\ell} > 0 \), and \( s \) be the number of terms for which \( p_{kj} - p_{k\ell} < 0 \). Then \( r \geq 1 \), and \( \bar{n} := r + s \geq 1 \) as well as \( \bar{n} \leq n \). More importantly

\[
\sum_k^- (p_{kj} - p_{k\ell})^+ = \sum_k^+ p_{kj} - \sum_k^+ p_{k\ell} \\
= 1 - \sum_k^- p_{kj} - \sum_k^+ p_{k\ell} \\
\leq 1 - s\delta - r\delta = 1 - \bar{n}\delta \\
\leq 1 - \delta < 1.
\]
The estimate for the difference \( M_i^{(t+1)} - m_i^{(t+1)} \) at last reduces to

\[
M_i^{(t+1)} - m_i^{(t+1)} \leq (1 - \delta)(M_i^{(t)} - m_i^{(t)}) \leq (1 - \delta)^t(M_i^{(1)} - m_i^{(1)}) \to 0,
\]
as \( t \to \infty \), showing \( M_i = m_i : = w_i \). As a consequence to the inequality \( m_i^{(t)} \leq p_{ij}^{(t)} \leq M_i^{(t)} \), we have \( \lim_{t \to \infty} p_{ij}^{(t)} = w_i \) for all \( j \). In matrix notation, \( \lim_{t \to \infty} P^t = [w, w, \ldots, w] \).

Next, we show the \( \lambda = 1 \) is an eigenvalue with eigenvector \( w \). In fact from the definition of \( w \) above \( \lim_{t \to \infty} P^t = [w, w, \ldots, w] \) and thus \( [w, w, \ldots, w] = \lim_{t \to \infty} P^t = P \lim_{t \to \infty} P^{t-1} = P[w, w, \ldots, w] = [Pw, Pw, \ldots, Pw] \) showing \( Pw = w \).

Next, we show the eigenvalue \( \lambda = 1 \) is simple. Let \( x \neq 0 \) be an eigenvector. Then \( Px = x \). Apply \( P \) to the identity repeatedly to have \( P^t x = x \). In limit, \( \lim_{t \to \infty} P^t x = [w, w, \ldots, w]x = (w_1 \sum x_j, w_2 \sum x_j, \ldots, w_n \sum x_j)^T = (x_1, x_2, \ldots, x_n)^T \). So \( x_i = w_i \sum x_j \) for all \( i \). Because \( x \neq 0 \), we must have \( \bar{x} := \sum x_j \neq 0 \), and that all \( x \) have the same sign. In other words, \( x = \bar{x}(w_1, \ldots, w_n)^T = \bar{x}w \) for some constant \( \bar{x} \neq 0 \), showing that the eigenvector of \( \lambda = 1 \) is unique up to a constant multiple. Finally, for any probability vector \( q \), the result above shows \( \lim_{t \to \infty} P^t q = (w_1 \sum q_j, w_2 \sum q_j, \ldots, w_n \sum q_j)^T = w \).

Next, let \( \lambda \) be an eigenvalue of \( P \). Then it is also an eigenvalue for the transpose \( P^T \). Let \( x \) be an eigenvector of \( \lambda \) of \( P^T \). Then \( P^T x = \lambda x \) and \( \|\lambda x\| = |\lambda||x| \leq \|P^T||x| \). Since \( \|P^T\| = 1 \) because \( \sum_{i=1}^n p_{ij} = 1 \) we have \( |\lambda| \leq 1 \).

Finally, let \( x \) be an eigenvector of an eigenvalue \( \lambda \). Then we have \( \lim_{t \to \infty} P^t x = W x = (\sum x_j)w \) on one hand and \( \lim_{t \to \infty} P^t x = \lim_{t \to \infty} \lambda^t x \) on the other hand. So either \( |\lambda| < 1 \) in which case \( \lim_{t \to \infty} \lambda^t x = 0 \) and then \( \sum x_j = 0 \), or \( |\lambda| = 1 \) in which case \( \lambda = e^{i\theta} \) for some \( \theta \) and the limit \( \lim_{t \to \infty} \lambda^t = \lim_{t \to \infty} e^{i\theta t} \) exists since \( \lim_{t \to \infty} e^{i\theta t} x = \lim_{t \to \infty} e^{i\theta t} \lambda^t x = \lim_{t \to \infty} P^t x = (\sum x_j)w \). The latter case holds if and only if \( \sum x_j \neq 0 \) and \( \theta = 0 \), i.e. \( \lambda = 1 \). This shows that all eigenvalues that is not \( \lambda = 1 \) are inside the unit circle.

References: Bellman(1977); Berman & Plemmons(1994); Frobenius(1908, 1912); Lancaster & Tismenetsky(1985); Marcus & Minc(1984); Perron(1907); Petersen(1983); Seneta(1973).