1. State and prove the Bolzano-Weierstrass Theorem. Explain clearly your use of any lemmas.

2. For each of the following statements, determine if it is true or false and provide either a proof or a counterexample, as appropriate.
   (a) For \( n \in \mathbb{N} \) and \( A \subseteq \mathbb{R} \), define \( A^n \) to be \( \{ a^n : a \in A \} \). If \( A \) is a bounded-above nonempty set of nonnegative real numbers, then, for \( n \in \mathbb{N} \), \( \sup A^n = (\sup A)^n \).
   (b) If \( (x_n) \) has the property that every subsequence of \( (x_n) \) has a convergent subsubsequence, then \( (x_n) \) converges.

3. If \( \lim_{n \to \infty} a_n = a \) and there are infinitely many terms of \( (a_n) \) which are greater than \( a \), then there is an decreasing subsequence of \( a_n \) which converges to \( a \).

4. Suppose the sequence \( (a_n) \) is decreasing and \( a_n - a_{n-1} > -1/n^2 \) for all \( n \in \mathbb{N} \). Prove that \( (a_n) \) converges.

5. Prove that every conditionally convergent series has a rearrangement that diverges to \(+\infty\), i.e., the sequence of partial sums diverges to \(+\infty\).

6. Suppose that \( (n_k) \) is a strictly increasing sequence of positive integers so that

\[
\lim_{k \to \infty} \frac{n_k}{n_1 n_2 \cdots n_{k-1}} = +\infty.
\]

Prove that \( \sum_{i=1}^{\infty} \frac{1}{n} \) converges to an irrational number.