(1) a) (10 points) Let $E \subset [0, 1]$ be a Lebesgue measurable set with $m(E) = 1$. Prove that $E$ is dense in $[0, 1]$.  
   b) (10 points) Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ and assume that $\mu$ is locally finite, i.e., for every $x \in \mathbb{R}$ there exists an open set $E \subset \mathbb{R}$ such that $x \in E$ and $\mu(E) < \infty$. Prove the $\mu(K) < \infty$ for every compact set $K \subset \mathbb{R}$.

(2) Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue measurable function and let $g_n(x) = (\sin f(x))^{2n}$, $x \in [0, 1]$, $n \in \mathbb{N}$.  
   a) Carefully explain why $g_n$ is Lebesgue measurable for every $n \in \mathbb{N}$. 
   b) Show all technical details in evaluating: $\lim_{n \to \infty} \int_{[0,1]} g_n(x) dm$.

(3) Prove that Fatou's Lemma and the Monotone Convergence Theorem are equivalent.

(4) a) Let $T : L^3(\mathbb{R}, m) \to \mathbb{C}$ be given by $T(f) := \int_{[0,\infty]} xf(x) dm(x)$. Prove that $T$ is a bounded linear functional on $L^3(\mathbb{R}, m)$ and find $\|T\|$. 
   b) Let $(X, \mathcal{M}, \mu)$ be a measure space. Give the definition of $L^\infty(\mu)$, and directly from the definition prove that: If $f, g \in L^\infty(\mu)$ then $f + g \in L^\infty(\mu)$ and $\|f + g\|_{\infty} \leq \|f\|_{\infty} + \|g\|_{\infty}$.

(5) Let $f : \mathbb{R} \to [0, \infty)$ be Borel measurable and define $\phi : (0, \infty) \to [0, \infty]$ by $\phi(y) = m\{x \in \mathbb{R} : f(x) > y\}$. Prove that: $\int_{\mathbb{R}} f(x)^2 dm(x) = 2 \int_{(0,\infty)} y \phi(y) dm(y)$.

(6) Let $X$ be a compact Hausdorff topological space with the following property:  
   If $E \subseteq X$ is open then its closure $\overline{E}$ is also open. Let $C(X)$ denotes the space of all continuous $\mathbb{C}$-valued functions on $X$. Given $f \in C(X)$ and $\epsilon > 0$, prove that there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $E_1, \ldots, E_n \subseteq X$ such that $\chi_{E_j} \in C(X)$ for every $j$ and $\sup_{x \in X} \left| f(x) - \sum_{j=1}^n \lambda_j \chi_{E_j}(x) \right| < \epsilon$.

(7) a) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f$ is continuous at $x_0 \in \mathbb{R}^n$. Prove that $x_0 \in L_f$, where $L_f$ is the Lebesgue set of $f$.  
   b) Consider the measure spaces $([0,1], \mathcal{B}_{[0,1]}, m)$ and $([0,1], \mathcal{B}_{[0,1]}, \mu)$, where $\mu$ is counting measure on $\mathcal{B}_{[0,1]}$. Prove that $m \ll \mu$, but $dm \neq f d\mu$ for any function $f$. Does this contradict the Radon-Nikodym Theorem? why?

(8) Let $f, f_n : [0, 1] \to \mathbb{R}$ such that $f, f_n \in L^2([0, 1], \mathcal{B}_{[0,1]}, m)$, for every $n \in \mathbb{N}$. Prove or disprove the following statements:  
   a) If $f_n \to 0$ a.e. $[0, 1]$ then $\|f_n\|_2 \to 0$.  
   b) If $f_n \to 0$ in $L^2([0, 1], m)$, then $f_n \to 0$ in measure.  
   c) If $f_n \to f$ weakly in $L^2([0, 1], m)$ and $\|f_n\|_2 \to \|f\|_2$, then $\|f_n - f\|_2 \to 0$.  
   d) If $\|f_n - f\|_2 \to 0$, then $\|f_n - f\|_p \to 0$ for every $p \in [1, 2]$.  

Real Analysis Comprehensive Examination–Math 921/922  
Friday, January 19, 2007, 2:00-6:00p.m., Burnett Hall 203  

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.  
   • Throughout the exam, the Lebesgue measure is denoted by $m$ and $\mathcal{B}_X$ denotes the Borel $\sigma$-algebra on a metric space $X$.  

• Write on one side of the paper only and hand your work in order.