# Polynomials and Intro to Complex Numbers • Summary • January 2011

(Note: many of the examples here were taken from *The Art of Problem Solving, Volume 2*, a resource for high school students interested in really digging into mathematical ideas, proving, and mathematical structures.)

## What does this have to do with the high school curriculum?

Michigan standards:

- algebra of polynomials (add, subtract, multiply, divide)
- know how to apply the
  - fundamental theorem of algebra
  - factor theorem
  - remainder theorem

  to find roots or zeros of polynomials.

**Today**: algebra of polynomials, factor theorem, statement of the fundamental theorem of algebra.

**Next time**: synthetic division, other tools for finding roots, complex numbers.

**Goal of the Day**: Understand the beautiful relationship between factoring, roots, and the degree of a polynomial.

## 1 Overview

**Definition.** *A* polynomial *is a function of the form*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

*where $n$ is finite, $a_0, a_1, \ldots, a_n$ are constants, and $a_n \neq 0$.*

*The* coefficients *of $f(x)$ are $a_0, \ldots, a_n$.*

Most of the time in this class, we will work with *real* polynomials: polynomials where $a_0, \ldots, a_n \in \mathbb{R}$. Sometimes, we will work with *complex* polynomials: polynomials where $a_0, \ldots, a_n \in \mathbb{C}$.

Similarly, most of the time when we work with polynomials (real or complex), we assume that the domain of the function is real – in other words, the $x$ that serve as the input for $f(x)$ is an element of $\mathbb{R}$. However, there will be times when we take $x \in \mathbb{C}$.

**Definition.** *The* degree *of $f(x)$ is $n$. We write* $\deg(f) = n$. **** Watch out! We will come back to this and revise it–slightly. See later in the notes. ****

**Definition.** *The $k^{\text{th}}$*-degree term *of a polynomial $f(x) = a_n x^n + \ldots + a_0$ is $a_k x^n$.*

**In this section, degree will be the most important feature of a polynomial!**

## 2 Adding polynomials

Polynomials are closed under addition and subtraction.

- What do you think happens to the degrees of polynomials when they are added together?
- Examples.

## 3 Multiplying and Dividing polynomials

- Polynomials are closed under multiplication.
- Are they closed under division?
- What do you think happens to degrees of polynomials when they are multiplied together?
  - it can go up. (example.)
  - can it go down?

    The answer is in the affirmative – it can. But the question is ... why? and what's an example?

  - can it stay the same? (yes: multiplying by constant.)

**Lemma** (Degree Lemma)**.** *Let $f(x)$ and $g(x)$ be two polynomials. Then*

$$\deg(fg) = \deg(f) + \deg(g).$$

*Proof.* [Our first proposed proof] Let $f(x) = a_n x^n + \ldots + a_0$, $g(x) = b_m x^m + \ldots + b_0$, where $a_n \neq 0$ and $b_m \neq 0$. Then $f(x)g(x)$ contains the term $a_n b_m x^{n+m}$, and this is the highest degree term of that polynomial: neither $a_n$ nor $b_m$ were 0, so $a_n b_m \neq 0$; and no other term can have a larger exponent. So $\deg fg = n + m$, which is $\deg f + \deg g$. □

- Our proof is pretty good, but it doesn't take into account one case. What is it? (When $f$ or $g$ is 0).
- Let $\deg(0) = -\infty$.
  - Sometimes the conventions of exceptional cases are defined to accommodate sensible lemmas rather than the most initially reasonable idea.
  - Cross off $\deg(0) = 0$ from above, replace with $\deg(0) = -\infty$.

*Proof.* [Our final proof] First let's address the case when neither $f$ nor $g$ are 0: let $f(x) = a_n x^n + \ldots + a_0$, $g(x) = b_m x^m + \ldots + b_0$, where $a_n \neq 0$ and $b_m \neq 0$. Then $f(x)g(x)$ contains the term $a_n b_m x^{n+m}$, and this is the highest degree term of that polynomial: neither $a_n$ nor $b_m$ were 0, so $a_n b_m \neq 0$; and no other term can have a larger exponent. So $\deg fg = n + m$, which is $\deg f + \deg g$.

Now let us address the case when at least one of the polynomials is 0. In this case, $fg = 0$, so $\deg fg = -\infty$. This satisfies the conditions of the proposition, as the product of $-\infty$ and any other degree is $-\infty$. □

## 4 More on Dividing Polynomials

- Long division by polynomials is very similar to long division of integers, especially if we remember that numerals are code for base 10.
- Compare

$$\begin{array}{r} 21 \text{ REM } 5 \\ 13\overline{)278} \end{array} \quad = \quad \begin{array}{r} 2\cdot 10 + 1 \text{ REM } 5 \\ 1\cdot 10 + 3\overline{)2\cdot 10^2 + 7\cdot 10 + 8} \end{array} \quad \text{with} \quad \begin{array}{r} 2x + 1 \text{ REM } 5 \\ x+3\overline{)2x^2 + 7x + 8} \end{array}.$$

A longer example:

$$
\begin{array}{r|l}
 & 2x^3 - 3x^2 + 5x - 4 \text{ REM } 8x+5 \\ \hline
x^2+2x+1 & 2x^5 + x^4 + x^3 + 3x^2 + 5x + 1 \\
 & -2x^5 - 4x^4 - 2x^3 \\ \hline
 & -3x^4 - x^3 + 3x^2 \\
 & 3x^4 + 6x^3 + 3x^2 \\ \hline
 & 5x^3 + 6x^2 + 5x \\
 & -5x^3 - 10x^2 - 5x \\ \hline
 & -4x^2 + 1 \\
 & 4x^2 + 8x + 4 \\ \hline
 & 8x + 5
\end{array}
$$

**Description of Long Division Algorithm**

- When dividing polynomials: multiply the divisor by whatever is necessary to cancel the highest degree term of whatever is leftover.
- This guarantees that the degree of whatever is leftover will always go down.
- When the degree of the leftover is less than the degree of the divisor, you have the final remainder and you are done.
- Since we have been subtracting successive multiples of the divisor, we know that

$$\text{dividend} = \text{divisor} \cdot \text{quotient} + \text{remainder}.$$

**Theorem** (Long Division Algorithm for Polynomials [Usiskin, Theorem 5.16])**.** *Let $a(x)$ and $b(x)$ be polynomials where $\deg(a) \geq \deg(b)$, and $b(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that*

$$a(x) = b(x)q(x) + r(x) \quad \text{and} \quad \deg(r) < \deg(b).$$

*We call $q(x)$ the* quotient *and $r(x)$ the* remainder.

*Proof.* The proof comes in two parts: (1) existence, (2) uniqueness.

**(1) Such $q(x)$ and $r(x)$ exist.** Let $q(x)$ be the quotient of $b(x)\overline{)a(x)}$, and let $r(x)$ be the remainder. Then $a(x) = q(x)b(x) + r(x)$ by Description of the Long Division Algorithm. It is also true that $\deg(r)$ must be less than $\deg(b)$, as the division algorithm must continue until this happens, again by Description of the Long Division Algorithm.

**(2) Such $q(x)$ and $r(x)$ are unique.** Suppose there are $q_1(x), q_2(x), r_1(x), r_2(x)$ such that

$$a(x) = b(x)q_1(x) + r_1(x) \text{ and } a(x) = b(x)q_2(x) + r_2(x).$$

Then

$$
\begin{array}{rcl}
a(x) & = & b(x)q_1(x) + r_1(x) \\
a(x) & = & b(x)q_2(x) + r_2(x) \\ \hline
0 & = & b(x)(q_1(x) - q_2(x)) + (r_1(x) - r_2(x)) \\
r_2(x) - r_1(x) & = & b(x)(q_1(x) - q_2(x))
\end{array}
$$

Hence $r_2(x) - r_1(x)$ is a multiple of $b(x)$. Note that $\deg(r_2(x) - r_1(x)) < \deg(b)$. The only way that the degree of a product can decrease is if one of the components is 0. Since $b \neq 0$, we must have $q_1 - q_2 = 0$. Hence $q_1 = q_2$. It follows that $r_1 = r_2$. We have shown uniqueness. ☐

# 5 The Relationship between Factors and Roots of Polynomials

The most common application of polynomials has to do with "solving" them. Here, the phrase "solving" means "finding roots."

**Definition.** *Let $f(x)$ be a polynomial. Then the* roots *or* zeros *of $f(x)$ are the solutions to the equation $f(x) = 0$.*

### Methods of finding roots

- factoring. if you can factor out an $(x - \alpha)$, it means $\alpha$ is a root.
- quadratic formula (only works for degree 2 polynomials). equivalent to breaking down quadratic polynomial into $(x - \alpha_1)(x - \alpha_2)$.
- plugging in and hoping $f(guess) = 0$ dividing by $x - guess$ and hoping remainder is 0.

All of these hinge on the following equivalence:

$$\text{[algebraic viewpoint]} \boxed{x - \alpha \text{ is a factor of } f(x)} \overset{\text{means}}{\longleftrightarrow} \boxed{f(\alpha) = 0} \text{[geometric viewpoint]}$$

> Thing to think about: the word "equivalence". In a mathematical context, "equivalence" means that they are conditions that guarantee each other. So to prove that two things are equivalent, you want to show that having either one can lead you to conclude the other. This means that proofs of equivalences have two parts.
>
> The reason that equivalences are important in math is that you can substitute equivalent things for each other. So if you are given a condition in a problem that $x - \alpha$ is a factor of $f(x)$, you could rephrase the problem with the condition that $f(\alpha) = 0$, and you'd end up with the same problem mathematically – and the problem might be easier to solve with the condition that $f(\alpha) = 0$.

**Theorem** (The Factor Theorem: A consequence of the Division Algorithm Theorem)**.** *Suppose $f(x)$ is a nonconstant polynomial, and $r$ is a constant. Then $\alpha$ is a root of $f(x)$* **if and only if** *$x - \alpha$ is a factor of $f(x)$.*

*Proof.* We must show two statements. First, we must show that $\alpha$ is a root implies $x - \alpha$ is a factor of $f(x)$. Second, we must show that $x - \alpha$ is a factor means $\alpha$ is a root.

**(1)** Suppose $\alpha$ is a root of $f(x)$, meaning $f(\alpha) = 0$. We show here that $x - \alpha \overline{)f(x)}$ has remainder 0, meaning $x - \alpha$ is a factor of $f(x)$.

Let $q(x)$ be the quotient for the division problem $x - \alpha \overline{)f}$. Since deg(remainder) $<$ deg$(x - \alpha) = 1$, we know that the remainder must be a constant. Let $r$ be the remainder. If $f(\alpha) = 0$, then

$$\begin{aligned} f(x) &= (x - \alpha)q(\alpha) + r \\ f(r) &= (\alpha - \alpha)q(\alpha) + r \\ 0 &= r \end{aligned}$$

**(2)** Suppose $x - \alpha$ is a factor of $f(x)$. Then there is a $q(x)$ such that $f(x) = (x - \alpha)q(x)$. Plugging in $\alpha$ yields

$$\begin{aligned} f(x) &= (x - \alpha)q(x) \\ f(\alpha) &= (\alpha - \alpha)q(\alpha) \\ f(\alpha) &= 0. \end{aligned}$$

We have shown $\alpha$ is a root. □

What is the big deal about the Factor Theorem? Because combined with the *Fundamental Theorem of Algebra*, this means that if we can use factoring to find all the roots!

**Theorem** (Fundamental Theorem of Algebra (FTA))**.** *Every nonconstant polynomial has at least one root, i.e., if $f(x)$ is a nonconstant polynomial, there is an $\alpha$ such that $f(\alpha) = 0$.* □

This $\alpha$ may be real, imaginary, rational, or irrational; whatever its nature, the Fundamental Theorem of Algebra assures us that a root exists. The proof is gorgeous as well as extremely intricate; it is provided as optional reading. We now use the Fundamental Theorem:

**Theorem** (The Degree $n$ Theorem: A consequence of the Factor Theorem and FTA)**.** *A nonconstant degree $n$ polynomial has $n$ roots. This means we can always represent a nonconstant polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ in the form*

$$f(x) = a(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n), \qquad (*)$$

*where $a$ is a constant, and the $\alpha_1, \ldots, \alpha_n$ are roots of $f(x)$.*

The form in $(*)$ is called the *factored form* of $f(x)$. (Quick comprehension question: how would you express $a$ in terms of the coefficients of the polynomials? And how would you express $a_0$ in terms of the roots of the polynomial?)

*Proof of the Degree n Theorem.* By the Fundamental Theorem of Algebra, $f(x)$ has at least one root. Call this root $\alpha_1$.

The Factor Theorem says that $\alpha_1$ is a root of $f(x)$ if and only if $x - \alpha_1$ is a factor of $f(x)$. So there is a polynomial $q_1(x)$ such that $f(x) = (x - \alpha_1)q_1(x)$.

If $f(x)$ was linear, then we are done. In this case $q_1(x)$ is a constant.

Otherwise, the polynomial $q_1(x)$ is a nonconstant polynomial of degree $n - 1$ and we repeat the process.

By the Fundamental Theorem of Algebra, $q_1(x)$ has at least one root. Call this root $\alpha_2$. By the Factor Theorem, there is a polynomial $q_2(x)$ such that $q_1(x) = (x - \alpha_2)q_2(x)$. Then $f(x) = (x - \alpha_1)(x - \alpha_2)q_2(x)$. The polynomial $q_2(x)$ has degree $n - 2$.

After $n$ steps, we get: $f(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)q_n(x)$, where $q_n(x)$ has degree $n - n = 0$. So it must be constant. Let $a = q_n(x)$. We have now shown the theorem. □

Now we can explain why the degree of a polynomial tells us how many roots the polynomial has – and that somewhere out there exists a factorization. Our proof hinged on the Factor Theorem and the Fundamental Theorem of Algebra, and each of those hinged on the Division Algorithm Theorem for Polynomials. We might ask about unique factorization. We can also use these theorems to prove that polynomials decompose uniquely into linear factors.

# 6 Methods for Finding Roots

## 6.1 Rational Root Theorem

Some of the greatest mathematicians have been stumped by this problem of finding roots in general, especially irrational ones. Luckily, if we're in the business of finding *rational roots*, we have a huge tool at our disposal. We will preview this to end the class.

**Preview.** Suppose that $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_n, a_{n-1}, \ldots, a_0$ are integers, and $\alpha$ is a rational root of $f(x)$. Then $\alpha$ is an integer.

Why: Because of the **Rational Root Theorem**: If $f(x) = a_n x^n + \ldots + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are integers, then all rational roots are of the form $p/q$, where if $p/q$ is in reduced form, then $p$ *must* be a factor of $a_0$ and $q$ *must* be a factor of $a_n$.

A related tool is ...

## 6.2 The relationship between roots and coefficients

If $r_1$ and $r_2$ are the roots of a quadratic polynomial $x^2 + bx + c$, then $b = r_1 + r_2$ and $x = r_1 r_2$. We can use this to figure out factorizations.

Another tool for finding roots is...

## 6.3 Figuring out that there are no negative roots

Suppose that our polynomial is

$$f(x) = 2x^3 - 5x^2 + 4x - 1.$$

What happens when we plug in a negative number?

Because all terms in the polynomial, that is

$$2x^3, \quad -5x^2, \quad ,4x, \quad \text{and } -1.$$

are negative when $x$ is negative, it means that

$$f(\text{negative number}) < 0,$$

so a negative number *cannot* be a root. So all of our roots must be positive.

We could now test the roots $1/2$ and 1. Testing 1 is easy, but testing $1/2$ is harder. Moreover, once we find roots, we'd like to simplify our work by only dealing with the quotient

$$q(x) = f(x)/(x - r).$$

How can we find $q(x)$ quickly?

## 6.4 Synthetic Division

We could use long division, but if the polynomial is huge, it could cause a headache! That is why we use synthetic division.

First I will demonstrate it using the

$$f(x) = 2x^3 - 5x^2 + 4x - 1,$$

and then we will talk about why it works. I will give two ways of thinking about it.

**Demonstration.** Is $\frac{1}{2}$ a root?

$$\begin{array}{c|rrrr} \frac{1}{2} & 2 & -5 & 4 & -1 \\ & & 1 & -2 & 1 \\ \hline & 2 & -4 & 2 & \boxed{0} \end{array}$$

The $\boxed{0}$ means that the remainder upon dividing by $x - \frac{1}{2}$ is 0. So, yes! $\frac{1}{2}$ is a root.

Moreover, the rest of the bottom line of this method tells us that

$$x - \frac{1}{2}\overset{\displaystyle 2x^2 - 4x + 2}{\overline{\big)\, 2x^3 - 5x^2 + 4x - 1}}$$

This is an amazingly fast way to do long division and test for roots! Does it always work? Why?

**Reason 1: Condensed Long Division.**

Let's examine what happens when we perform the long division $x - \frac{1}{2}\overline{\big)\,2x^3 - 5x^2 + 4x - 1}$.

$$\begin{array}{r r}
 & 2x^2 - 4x + 2. \\
\cline{2-2}
x - \frac{1}{2}\big) & 2x^3 - 5x^2 + 4x - 1 \\
 & -\,2x^3 \;\; + x^2 \qquad\qquad \\
\cline{2-2}
 & -\,4x^2 + 4x \qquad \\
 & 4x^2 - 2x \qquad \\
\cline{2-2}
 & 2x - 1 \\
 & -\,2x + 1 \\
\cline{2-2}
 & 0
\end{array}$$

**Reason 2: Convenient Algebra.**

(See explanation in Usiskin.)

So when we are dividing a polynomial by a first-degree polynomial, the synthetic division algorithm is *equivalent* to long division!

## 6.5 Example that uses everything!

Find all the solutions of the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$$

By the Rational Root Theorem, all the rational roots must be integer factors of 24. So we should first test those factors and see how far we can get before looking for non-rational roots.

When we plug in a negative number, we find that all the terms of the polynomial are negative. Hence all the roots are positive.

Let us test if 1 is a root using synthetic division. (The number 1 is always a good place to start.)

$$\begin{array}{r|r r r r r}
1 & 1 & -10 & 35 & -50 & 24 \\
 & & 1 & -9 & 26 & -24 \\
\hline
 & 1 & -9 & 26 & -24 & \boxed{0}
\end{array}$$

It is a root! We now know that

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = (x - 1)(x^3 - 9x^2 + 26x - 24).$$

Let us test if 2 is a root of $x^3 - 9x^2 + 26x - 24$.

$$\begin{array}{r|rrrr} 2 & 1 & -9 & 26 & -24 \\ & & 2 & -14 & 24 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}.$$

We have now computed that

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = (x-1)(x-2)(x^2 - 7x + 12),$$

which we may now factor into:

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = (x-1)(x-2)(x-3)(x-4).$$

# 7 Intro to Complex Numbers

## 7.1 What are Complex Numbers?

- Complex numbers are roots of real polynomials. Examples: Solutions to $x^2 + 1 = 0$, $x^2 + x + 2$.
- An example of a complex number is $i$, which stands for $\sqrt{-1}$.
- In general, complex numbers are numbers with the form $a + bi$.
- We say that the *real part* of a complex number $a + bi$ is $a$, and the *imaginary part* is $b$. We write $\mathrm{Re}(a + bi) = a$, $\mathrm{Im}(a + bi) = b$. Caution! The imaginary part is the *coefficient* of $i$. The imaginary part is not an imaginary number.

The goal of learning complex numbers today is to prove the following theorem.

**Theorem** (Conjugate Pair Theorem ). *If $f(x)$ is a real polynomial and $a + bi$ is a root of $f(x)$, then $a - bi$ must also be a root of $f(x)$.*

On the way to proving this theorem, we will discuss

- visualizing complex numbers
- adding and subtracting complex numbers
- multiplying complex numbers
- conjugate pairs for complex numbers.

## 7.2 Visualizing complex numbers

How can we visualize this? Take inspiration from the number line. Need two number lines, one for $i$ and the other for the real part. Get the complex plane, one point per complex number.

This visualization gives us another way to view complex numbers: as vectors.

For example, the complex number $(5 - 4i)$ can be represented as the vector $(5, -4)$.

In general, the complex number $a + bi$ can be represented as the vector $(a, b)$.

Sometimes we may want to just use one letter to represent a complex number. For example, suppose we let $z$ be a complex number. Then we write down the real and imaginary parts of $z$ using $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. Using vector notation, $z$ can be represente In other words, if $z$ is a complex number, then its vector is $(\mathrm{Re}(z), \mathrm{Im}(z))$.

## 7.3 Adding and subtracting complex numbers

**Algebraic viewpoint.** We add complex numbers

$$a + bi, \quad c + di$$

by adding the real parts and imaginary parts separately:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Example. $(4 + 5i) + (-7 + i) = (4 - 7) + (5 + 1)i = -3 + 6i$.

**Geometric viewpoint.** Adding complex numbers

$$a + bi, \quad c + di$$

is like adding the vectors

$$(a, b), \quad (c, d),$$

because $(a, b) + (c, d) = (a + c, b + d)$.

## 7.4 Multiplying complex numbers

When multiplying complex numbers

$$a + bi, \quad c + di$$

we use distributivity:

$$\begin{aligned} (a + bi)(c + di) &= ac + bci + adi + bdi^2 \\ &= ac - bd + (bc + ad)i. \end{aligned}$$

- Suppose you are given the complex number $5 - 4i$. What complex number could you multiply this by to get a real number? (Trivial answer: 0. Better answer: $5 - 4i$ or any multiple of it.)
- What about the complex number $100 + 302i$? (Answer: $100 - 302i$).
- Why does multiplying $a - bi$ by $a + bi$ always give a real number? (Because of difference of squares: the imaginary terms cancel out during the expansion.)

## 7.5 Conjugate pairs

The examples we just saw are called *conjugate pairs*.

If $z = a + bi$, then we write $\overline{z}$ to mean $a - bi$.

Why the conjugate is useful:

- You can multiply $z$ by $\overline{z}$ to get a real number.
- You can add $z$ to $\overline{z}$ to get a real number.

Why the conjugate is beautiful and useful:

- $\overline{z} \cdot \overline{w} = \overline{zw}$.

- $\overline{z} + \overline{w} = \overline{z + w}$.
- $\overline{az} = a\overline{z}$ if $a$ is real, and $\overline{a} = a$ if $a$ is real.
- The conjugate of the conjugate is the original number:

$$\overline{\overline{z}} = z.$$

- The average of a complex number and its conjugate is the real part.

$$\frac{(a+bi)+(a-bi)}{2} = a$$

- Multiplying a complex number with its conjugate gives you a way to find the geometric size of the complex number.

  The length of the vector representation of $a + bi$ is $\sqrt{a^2 + b^2}$ by the Pythagoras Theorem. So this means that $(a + bi)(a - bi)$ is the square of the length!

## 7.6 The Conjugate Pair Theorem

**Theorem** (The Conjugate Pair Theorem)**.** *If $f(x)$ is a real polynomial and $a + bi$ is a root of $f(x)$, then $a - bi$ must also be a root of $f(x)$.*

Before doing the proof, let's do an example. Suppose that we know that $a + bi$ is a root of the polynomial

$$f(x) = 2x^3 + x + 10.$$

We want to show that $a - bi$ is also a root. In other words, we want to show that $f(a + bi) = 0$ implies that $f(a - bi) = 0$.

Let us try to find $f(a - bi)$. To make the notation simpler, let's write $z = a + bi$, so $\overline{z} = a - bi$. Then

$$\begin{aligned} f(\overline{z}) &= 2\overline{z}^3 + \overline{z} + 10 \\ &= 2\overline{z^3} + \overline{z} + 10 \\ &= \overline{2z^3} + \overline{z} + \overline{10} \\ &= \overline{2z^3 + z + 10} \\ &= \overline{f(z)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$

We have shown that if $a + bi$ is a root of $2x^3 + x + 10$, then so must its conjugate! This proof did not exploit many specific properties of the particular polynomial, so we can use it to guide the general proof.

*Proof of the Conjugate Pair Theorem.* Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, and suppose that $z = a + bi$ is a root of $f(x)$. Then $f(z) = 0$.

We want to show that $f(\overline{z}) = 0$.

$$\begin{aligned} f(\overline{z}) &= a_n\overline{z}^n + a_{n-1}\overline{z}^{n-1} + \ldots + a_2\overline{z}^2 + a_1\overline{z} + a_0 \\ &= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \ldots + a_2\overline{z^2} + a_1\overline{z} + a_0 \\ &= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \ldots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} \end{aligned}$$

$$\begin{aligned}
&= \overline{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0} \\
&= \overline{f(z)} \\
&= \overline{0} \\
&= 0.
\end{aligned}$$

So $\bar{z} = a - bi$ *must* be a root of $f(x)$. □

Now we have another tool to our root-finding arsenal: if we know that a real polynomial has a complex root, we know that the conjugate of the complex root is also a root.

In fact, the proof technique can be used to give us something more: suppose we know that $a - \sqrt{b}$, where $\sqrt{b}$ is irrational, is a root of a polynomial with rational coefficients. Then in much the same way, we can show that $a - \sqrt{b}$ must also be a root of this polynomial.