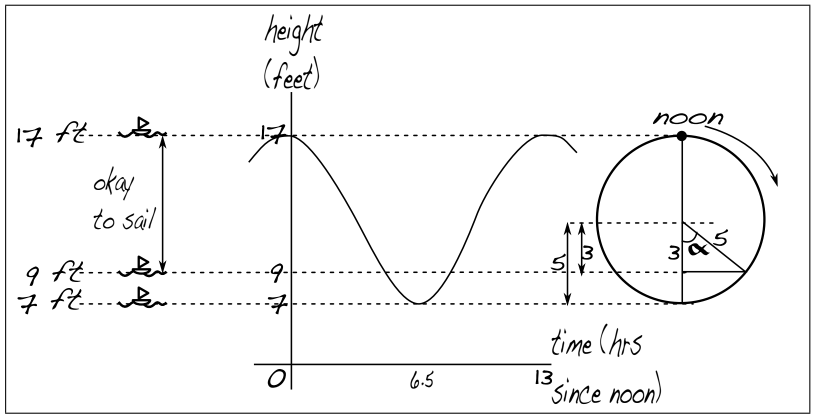
Problem Set 5

1. You've assigned the following problem to your students.

   > A sailboat is docked at noon, when it is high tide. For that week, high tide is approximately 17 feet, low tide is approximately 7 feet, and there are approximately 6.5 hours between high and low tide. To sail safely, the tide should be at least 9 feet. Assume you can model tidal movement as a sinusoidal function. What is the last time in the afternoon that the boat can set sail? Your answer should be correct to the minute.

   One student's solution shows the following representation of the problem set up:



   Later in the student's work, you see:

   The last time that they can set sail is:

   $$12\text{P.M.} + \frac{\pi - \alpha}{\pi} \times (\quad) \approx 4{:}35\text{P.M.}$$

   a) Assuming that it is mathematically correct, what was smudged out?
      - (A) $2\pi \times 5$, because that's the circumference of the circle in feet, the unit used in the problem for tide height.
      - (B) 13, because that's the tide cycle time in hours, the unit used in the problem for time.
      - (C) $\pi \times 5$, because that's the half the circumference of the circle in feet, the unit used in the problem for tide height.
      - (D) 6.5, because that's half the tide cycle time in hours, the unit used in the problem for time.

      Connect your answer to an explanation of what proportion $\frac{\pi-\alpha}{\pi}$ represents in the context of the problem (movement of height time, time, sailing).

   b) What does the arrow next to "noon" in the circle schematic represent? (What does the direction represent? If the dot is moving along the circle in the direction of the arrow, what does the dot represent in the context of the problem?)

   c) An alternate solution to this problem is to use the sinusoidal function which models the movement of the tide height H with respect to time t:

   $$H(t) = 12 + 5\cos\left(\frac{\pi}{6.5}t\right). \qquad (*)$$

   (continued on next page)

A student uses this equation and finds:

$$12 + 5\cos(\frac{\pi}{6.5}t) = 9 \rightarrow \cos(\frac{\pi}{6.5}t) = -\frac{3}{5} \rightarrow t = \frac{6.5}{\pi}\arccos(-\frac{3}{5}) \approx 4.581\ldots$$

Since 0.581 *hours is* $\approx$ 35 *minutes, the last time in the afternoon that the vacationers can safely sail is 4:35pm.*

In the breakroom later that day, you show this work to another teacher, who says: "If you had written a problem where the starting time had been 1:00pm, I wonder if student would have still answered 4:35pm, instead of 5:35pm. I think that your student might have exploited a coincidence to get the correct answer." What coincidence is this teacher referring to? What answer would the student have gotten with the suggested shift in high and low tide numbers? What does the 4.581... in the student's answer really represent?

d) You are designing an in-class problems to follow up on the original problem. You are deciding between the two following choices:

(A) What is the last time between 6:30pm and midnight that it is safe to sail?

(B) What is the first time between 6:30pm and midnight that it is safe to sail?

You decide to opt for the choice that requires more calculation. (Which one is that?) Solve the problem.

e) The student in problem (c) comes up to you (after you and she have already had a good conversation about the issue that the breakroom teacher brought up). Now the student confused about why her solution strategy doesn't work for your follow-up problem – she is only able to find the time 4:35pm and no other time. You decide that you should discuss with the class how to find all possible solutions to the equation (*) in part (c). Find all possible solutions.

f) You are designing yet another in-class problem to follow up on the original problem. You want to use the same set up as the original (same low tide, high tide, time between low and high tides, etc.), except for changing what the problem states as the minimum tide height necessary for safe sailing.

(i) Suppose you change the height to 10 feet. Using the same representation as the student did, what would α be in this case? (You may express it as an arccos).

(ii) Is it possible to choose an integer for the minimum tide height and have α be a special angle?

2. At the end of these pages, you will find a table (spanning two sides of a page). Complete the table. The graphs you produce for the table should give a sense of the function. In particular, they should have the following features:

- They are hand drawn.
- If the function is periodic, the graph should show at least 3 periods
- scales on the input and output axes should be given that locates key points of the function's graph (e.g., minima, maxima, boundary points, periodicity).
- The proportion of the scales on the input and output graph should be 1:1; that is, if you were to draw a slope of 1 on your axes, it would make a 45° angle with the horizontal axis.
- Unless specially noted, all angles should be be taken in radians.

(problem set continues next page)

3. Let A be a set of real numbers. Consider these statements.

Statement I: Every element of $\mathbb{R}$ is a limit of a subset of numbers in A. (Given any real number, you can find a sequence of numbers from A that converge to that real number.)

Statement II: Between any two points of $\mathbb{R}$, there is a point of A.

Statement III: Between any two points of A, there is a point of A.

a) Determine to your best ability, for the following sets, whether the above statements are true or false.

A	**I**: true or false?	**II**: true or false?	**III**: true or false?
rational numbers			
irrational numbers			
$\{a$ is a real number in the open interval $(n, n+1) \mid n$ is an integer$\}$			
$\{a$ is a real number in the open interval $(n, n+1) \mid n$ is even$\}$			
$\{a$ is a real number in the half closed/open interval $(n, n+1] \mid n$ is even$\}$ ("clopen" intervals)			
$\{a$in the closed interval $[n, n+1] \mid n$ is even$\}$			
"clopen" interval $[0, \infty)$			

b) Two of the statements describe equivalent conditions.

i. Recall in a tweet-sized description what it means for two conditions to be equivalent.
ii. Which two statements are equivalent?
iii. Explain why they are equivalent.

Name:____________________________

Some intriguing compositions of trig functions…

function	domain, represented as a set of real numbers	image, represented as a set of real numbers	does the domain correspond to *angles*, *x-coordinates*, or *y*-coordinates of the unit circle?	does the image correspond to *angles*, *x-coordinates*, or *y*-coordinates of the unit circle?	graph (instructions under Problem 2)
$f(t) = \sin(\arcsin(t))$					
$f(t) = \arcsin(\sin(t))$					
$f(t) = \arcsin(\cos(t))$					
$f(t) = \sin(\arccos(t))$					

Name:____________________________

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$f(t) = \cos(\arcsin(t))$					
$f(t) = \sin(t)$ and $g(t) = \sin(\frac{\pi}{2}\sin(t))$ graphing both in the same coordinate plane					
$f(t) = \sin(\frac{\pi}{2}\sin(t))$, where t is taken in *degrees*					
$f(t) = \arcsin(\sin(t))$, where t is taken in *degrees*					