Problem Set 4

Relevant concepts

- Polynomial division and polynomial roots
  - Rational Root Theorem
  - Relationship between roots and coefficients
  - Synthetic division
- Complex numbers
  - Definition of complex number
  - Complex number arithmetic
  - Representation of complex numbers as vectors
  - Definition and properties of complex conjugate pairs
- Polynomials and conjugates: the Conjugate Pair Theorem.

Problems

1. a) Find all roots of $x^4 + x^3 + 2x^2 + 17x - 21$. You may need to use the quadratic formula at some point for this problem.
   b) Find all the roots of $x^3 - 7x^2 + 14x - 8$.
   c) Find all the roots of $6x^3 + 17x^2 + 11x + 2$. Before testing roots, it may be helpful to answer the question: how many of the roots can be negative? How many can be positive?
2. True or False:

$$b(x)\overline{\big)a(x)}^{\,q(x)\text{REM } r(x)} \qquad \text{implies} \qquad q(x)\overline{\big)a(x)}^{\,b(x)\text{REM } r(x)}.$$

   In other words, if dividing $a(x)$ by $b(x)$ gives a quotient of $q(x)$ and a remainder of $r(x)$, then dividing $a(x)$ by $q(x)$ gives a quotient of $b(x)$ and the same remainder.
   If true, give a proof. If false, give a counterexample.
3. a) This problem examines numbers of the form $a + b\sqrt{7}$, where a and b were rational. Let $\overline{a + b\sqrt{7}}$ denote $a - b\sqrt{7}$, and call it the *conjugate* of $a + b\sqrt{7}$.
   We can use the terminology $\mathbb{Q}\langle\sqrt{7}\rangle$ to denote the set of all numbers of the form $a + b\sqrt{7}$, where a and b are rational. In the following, suppose $\alpha, \gamma \in \mathbb{Q}\langle\sqrt{7}\rangle$.
      i. Find $\overline{\overline{\alpha}}$.
      ii. Find $\overline{\alpha + \gamma}$ in terms of $\overline{\alpha}$, $\overline{\gamma}$.
      iii. Find $\overline{\alpha\gamma}$ in terms of $\overline{\alpha}$, $\overline{\gamma}$.
      iv. What is $\overline{a + b\sqrt{7}}$ when $b = 0$?
      v. Explain why, if $\alpha \in \mathbb{Q}\langle\sqrt{7}\rangle$ is a root of a polynomial $f(x)$, then $\overline{\alpha}$ must be a root of that polynomial as well! Present your explanation in proposition/proof format.

Fact to think about and potentially use in #4: if we replaced $\sqrt{7}$ with any other irrational square root of an integer, we'd actually get yet another conjugate pair theorem! So this idea is one that has a lot of power.

4. A quartic polynomial $f(x)$ has rational coefficients, constant term -1, and roots $3-i$ and $1+\sqrt{2}$. Find $f(x)$. You may express it as a product of factors. Some questions that may be useful to think about on the way to finding $f(x)$ are: how many roots does $f(x)$ have? What do those roots have to be? What is the relationship between roots and the formula of $f(x)$? How can we express the constant term using roots?

   Aside. Some potentially useful vocabulary: *quadratic* means degree 2, *cubic* means degree 3, *quartic* means degree 4, and *quintic* means degree 5.

5. a) Find the roots of x^3+1. You will get three roots, two of which are not real.
   b) All the roots are complex, and may be represented as vectors in the the complex plane. Find the length of each of the vectors using the Pythagoras Theorem.