Problem Set 3

Relevant concepts

- Periodicity of decimal expansions for rational numbers (*Feasting on Leftovers*, on cTools)
- Long division algorithm (Usiskin, p. 205)
- Division Algorithm Theorem (Usiskin, p. 206)

Terminology

The *leftover sequence* for a division problem such as $7\overline{)2}$ is the sequence of terms used in the traditional long division algorithm to generate the decimal digits. For every decimal digit, there is an associated term in the leftover sequence.

The leftover sequence for $7\overline{)2}$ consists of repeated blocks:

$$\hookrightarrow 2 \to 6 \to 4 \to 5 \to 1 \to 3 \hookleftarrow .$$

They appear in the long division work as bolded on the work shown to the right.

```
      .2 8 5 7 1 4 ...
  7 ) 2.0
      1.4
        6 0
        5 6
          4 0
          3 5
            5 0
            4 9
              1 0
                7
              3 0
              2 8
                2 0
                etc. ...
```

The *decimal sequence* for a division problem is the sequence of its decimal digits. The decimal sequence for $7\overline{)2}$ consists of the repeated block: $\hookrightarrow 2, 8, 5, 7, 1, 4 \hookleftarrow$.

Problems

1. a) Find the leftover sequences for the division problems
      i. $50\overline{)1}$
      ii. $7\overline{)3}$
      iii. $12\overline{)7}$
      iv. $9900\overline{)56}$.

      Show how to find the leftover sequences *without* using the standard representation of long division. (See Section 5 of *Feasting on Leftovers*.) For $50\overline{)1}$, you should find 3 distinct numbers in the leftover sequence. For $9900\overline{)56}$, you should find 4 distinct numbers in the leftover sequence.

   b) For each fraction, show how to obtain the decimal expansion from the leftover sequences. What do your leftover sequences tell you about the size of the repeating blocks of the decimal expansions of these fractions?

2. You are told that the leftover sequence of a division problem $b\overline{)1}$ is

$$1 \to 1 \to 1 \to \ldots$$

   Find all the possible b. What are the decimal expansions of the fractions $\frac{1}{b}$?

3. The decimal expansions for the fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, and $\frac{6}{7}$ could all be described as "rotations" of each other.

   a) In your own words, give a definition for what "rotation" means in this context. Circle or otherwise highlight the conditions of your definition.
   b) Explain, using leftover sequences, why the decimal expansions satisfy the conditions of your definition of "rotation". Express your explanation in proposition/proof format.

4. Find the leftover sequences for $\frac{1}{13}$ and $\frac{2}{13}$.

   a) Find the positive integers n such that $\frac{n}{13} < 1$ and the decimal expansion of $\frac{n}{13}$ shares the same digits as the repeating decimal expansion of $\frac{1}{13}$. Explain how you know your list of possible n is complete.
   b) Are there n such that the decimal expansion of $\frac{n}{13}$ is not a rotation of either $\frac{1}{13}$ or $\frac{2}{13}$? Prove your assertion. Write your solution with a proposition/proof format. Make sure to explain in your proof how to obtain the decimal expansion from the leftovers.

The questions in this box are interesting queries related to #4 that you are highly encouraged to talk to each other about, especially if you are doing homework in groups (which you are also strongly encouraged to do). You do not have to turn these in. However, thinking about them and talking about them may help sharpen your understanding of the concepts addressed by question #4.

- The numbers in the leftover sequence for $13\overline{)1}$ and $13\overline{)2}$ do not overlap, but the numbers in the decimal expansions for $\frac{1}{13}$ and $\frac{2}{13}$ do. Why is it okay for decimal expansions to have numbers in common even when the leftover sequences have no numbers in common?
- What are the digits that the decimal expansions for $13\overline{)1}$ and $13\overline{)2}$ have in common? What leftovers did they come from in each case?

5. (Factorization concepts – submit online sometime over the weekend of Jan 29-30 at `http://tinyurl.com/4jpzduj`) We discussed in class the difference between the ideas:

   - "The number of times that 7 *goes into* 21 is three"
   - "The *number of factors of* 7 in 98 is 2."

   a) Write a definition for *number of factors of* that would be suitable for a high school student comfortable with exponents.
   b) Write a definition for *number of factors of* that does not use any exponents, and would be suitable for a student in eighth grade.

   One potential minefield to look for is making sure that your definition doesn't imply that the "number of factors of" could be less than it should be. For example, your definition should rule out the (false) idea that "the number of factors of 7 in 98 is 1." Check that this false statement does *not* satisfy the conditions of your definition.

Fun follow-up fact. Someone asked about the word "lemma" in class. Here is what the Oxford English Dictionary had to say.
Plural form: lemmas, lemmata.
Etymology: From the Greek $\lambda\acute{\epsilon}\mu\mu\alpha$, derived from $\lambda\acute{\epsilon}\pi$-$\epsilon\iota\nu$, meaning "to peel."
Definitions: (In *Math.*, etc.) A proposition assumed or demonstrated which is subsidiary to some other.
(Other usages:) The husk or shell of a fruit; the primary or outer layer of a germinal vesicle; in grasses, the lower bract of a floret.