Math486-W11: Content Course for Future Secondary Teachers

Course Meeting Time: Monday and Wednesday 7-8:30pm, East Hall 1084

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Overview

How do definitions and representations influence the ways you think about and use mathematical concepts? Do different representations make different mathematics visible? What does it mean to speak in mathematically responsive ways with high school students, and how might this be different than simply being friendly or being mathematically precise? What considerations about language, explanation, and representations arise in teaching secondary content? What structures are used in communicating, doing, and explaining mathematics?

These are the sorts of questions that this course will explore as we uncover the mathematical entailments of questions such as –

- Suppose someone tells you that \( \sqrt{3} + \sqrt{7} - \sqrt{8} - 2\sqrt{7} \) is a familiar rational number. How would you identify this rational number? Is it important in solving for the number that \( \sqrt{7} \) is irrational?
- You may know that fractions can be converted into decimals that either terminate or eventually repeat. In fact, the block of repeating numbers can be no bigger than the denominator. Is there a good explanation for this? Could we find the possible sizes of blocks for a given denominator?
- Why is it that reflecting the graph of an invertible function \( f(x) \) yields the graph of its inverse function?

At the end of this course, you should be able to discuss features of and deliver good explanations of key secondary mathematical content, evaluate the affordances of various representations of common families of functions, and discuss the features of and choose mathematically careful notation and definitions.
This course is one of a triplet of required courses for future secondary mathematics teachers. This is the content course; its sibling courses are the methods course (ED 413) and literacy course (ED 402). The courses making up this triplet are meant to complement each other: you are welcome – in fact strongly encouraged – to bring and use knowledge from one class to the other!

**Course Modules**

Math486-W11 is split into five modules. Each module covers a swath of the secondary mathematical terrain – from the perspective of one who must teach it. The goals for each module can be thought of in terms of know-how that supports the work of teaching as well as in terms of the discipline of mathematics.

Themes underlying the goals supporting the work of teaching include explanation, language (including notation), representations, definitions, and mathematical analysis (including of conditions, conclusions, and structures).

To underscore the emphasis on explanation, this course’s mathematical goals have been inspired by the notion of an “instructional explanation”. You should be able to explain and use the content we learn, in a mathematically careful way, with well-chosen notation and/or families of examples.

Two of the goals will be selected for assessment on a performance exam, one in the middle of the term, and one at the end of the term. A rubric will be given to you at least two weeks before the practical exam.

**I. Conditions and Conclusions: Primes, Rationals, Irrationals, and Radicals**

Much of the work in this module supports the Michigan Merit Curriculum Content Expectations for Geometry, in particular Standard L3. (You can find an excerpt of these standards later in this syllabus.)

**Supporting the Work of Teaching**

At the end of this module, you should be able to:

- Recognize the multiple ways in which the concept of a “condition” may arise in the context of secondary mathematics especially in the statements of mathematical results, in the intermediate stages of mathematical discovery, and in word problems; identify and use conditions in these cases.
- Discuss, using mathematically careful language, how the conditions of a particular task affect the possible conclusions; identify and describe which conditions have been used in the course of a solution and how they have been used.
- Discuss and recognize language considerations that arise in the context of factoring and divisibility; and in using the words “and” and “or” when describing conditions or conclusions; correctly identify and use the mathematical conventions in these contexts.
- Understand in what sense “if and only if” denotes the equivalence of two conditions.
- Identify some features of a good explanation of a mathematical result or concept, including features of mathematically careful language.
Mathematical Knowledge
At the end of this module, you should be able to explain and use, in a mathematically careful way, with well-chosen notation and/or families of examples:

- Arithmetic with rational and irrational numbers, including the correct completion of the table below.
- Euclid’s Lemma as a consequence of Unique Prime Factorization, including explaining why the common errors that arise in applying Euclid’s Lemma are errors, in terms of conditions and conclusions. (Euclid’s Lemma states that if \(a\) and \(b\) are two integers, and a prime \(p\) is a factor of \(ab\), then \(p\) must be a factor of \(a\), a factor of \(b\), or a factor of \(a\) and a factor of \(b\).)
- Proofs by contradiction, including their structure and how conditions and conclusions of the intended result fit into the structure; using proof by contradiction to prove the irrationality of \(n\)th roots (for numbers that are not perfect \(n\)th powers). Identifying when a proof by contradiction has been used implicitly.
- Use set notation to describe collections of numbers such as \(\mathbb{Q}(\sqrt[3]{7}) = \{a + b\sqrt[3]{7} \mid a, b \in \mathbb{Q}\}\) or \(\mathbb{Q}(\sqrt{7}) = \{a + b\sqrt{7} \mid a, b \in \mathbb{Q}\}\); and describe the structure of such sets in terms of arithmetic closure and uniqueness of coefficients. (This will help you when learning about polynomials!)
- Use the proposition/proof format to present a mathematical result.

<table>
<thead>
<tr>
<th>the sum of …</th>
<th>the product of …</th>
</tr>
</thead>
<tbody>
<tr>
<td>a rational and a rational is</td>
<td>a rational and a rational is</td>
</tr>
<tr>
<td>(a) always rational</td>
<td>(a) always rational</td>
</tr>
<tr>
<td>(b) could be rational or irrational</td>
<td>(b) could be rational or irrational</td>
</tr>
<tr>
<td>(c) always irrational</td>
<td>(c) always irrational</td>
</tr>
</tbody>
</table>

II. Solutions and Representations: Polynomials and their Roots, Complex and Real Roots, and Decimals

Supporting the Work of Teaching
At the end of this module, you should be able to:

- Recognize the multiple ways in which the concept of a “solution” may arise in the context of secondary mathematics, how they relate, and their affordances in terms of making mathematical content visible.
- Evaluate the affordances of multiple representations of polynomials and their roots, map between these representations, and discuss the mathematical concepts that these representations may highlight or obscure.
- Evaluate the affordances of alternative definitions of the degree of a polynomial.
• Discuss and recognize language considerations that may arise when discuss-
ning “roots”, “factors”, and “solutions” of polynomials.

Mathematical Knowledge
At the end of this module, you should be able to explain and use, in a mathemati-
cally careful way, with well-chosen notation and/or families of examples:

• The relationship between decimal representation of rational numbers, frac-
tion representation of rational numbers, and the long-division algorithm; and the crucial role of the denominator.
• Mathematical parallels between the long-division algorithm for integers and the long-division algorithm for polynomials – and where using the parallels requires caution.
• Why degree is a critical feature of a polynomial.
• The relationship between the long-division algorithm for polynomials, the synthetic division algorithm for polynomials, roots of a polynomial, and factors of a polynomial.
• The mapping between geometric and algebraic viewpoints on factors and roots.
• Why complex roots of real polynomials come in pairs, and why this extends to roots from subsets of reals such as \( \mathbb{Q}(\sqrt{7}) \).

III. Coordinating Representations and Definitions: Trigonometric Functions, Functions, and Inverses

Supporting the Work of Teaching
At the end of this module, you should be able to:

• Discuss and recognize language considerations that arise in the context of using and defining inverse functions, especially the inverse sine and inverse cosine functions and notation.
• Coordinate between graphical, algebraic, verbal, and tabular representations of functions; evaluate the affordances of these representations in terms of the mathematics they make visible.
• Coordinate between the unit circle definition of sine and cosine, the graphs \( y = \sin(x), y = \cos(x) \), and a diagram of the movement of a spring (or other object that exhibits sinusoidal but non-rotational movement).
• Recognize and discuss the language considerations that arise in the context of using the unit circle definition of sine and cosine, for example, in using the phrase “half way up”.
• Evaluate the mathematical demand of various “ferris wheel problems” using sine, cosine, and their inverse functions.
• Define and discuss the definition of the inverse sine and inverse cosine functions in terms of their graphs, the unit circle definition, tabular representations, and algebraic representations; explain in what sense they are inverses.

Mathematical Knowledge
At the end of this module, you should be able to explain and use, in a mathemati-
cally careful way, with well-chosen notation and/or families of examples:
The definition of function, invertibility, and inverse function, in terms of graphical, verbal, tabular, and algebraic representations.

The definition of the sine and cosine functions, their inverses; and common trigonometric identities.

“Ferris wheel problems” and “spring problems”.

Common errors in using the sine function, cosine function, and their inverses; why they might arise, and why they are errors, explained in terms of the definition of inverse function and the notations commonly used. (For example, it is not always true that \( \sin^{-1} \sin(x) = x \).)

IV. Representations, Families, and Definitions: Sequences of Decimals, Complex Numbers, and Polynomials

Supporting the Work of Teaching
At the end of this module, you should be able to:

- Discuss and recognize language considerations that arise, especially with regard to notation, when explaining, using, and stating DeMoivre’s rule; and in the context of finding the solutions of equations such as \( z^n = w \) or \( (az + b)^n = w \).
- Explain the notion of limits, convergence, and divergence in mathematically careful language; apply this notion to limits of sequences of decimals, complex numbers, and polynomials; explain these notions in terms of the various representations of real numbers (points on the real line, decimal), complex numbers (polar, cartesian, graphical) and polynomials (graphical, expanded form, factored form).
- Discuss and recognize common errors that arise in defining limits.
- Coordinate between different representations of and the most common notations for complex numbers, including cis, polar, cartesian, and vector notations.
- Define complex numbers in terms of different representations; evaluate the affordances of alternative definitions.
- Discuss and recognize pervasive ways of thinking about decimal representation of real numbers that may impact learning of limits.

Mathematical Knowledge
At the end of this module, you should be able to explain and use, in a mathematically careful way, with well-chosen notation and/or families of examples:

- Complex numbers and the complex plane, including polar form, cartesian form; angles and magnitudes of complex numbers, including the “special angles”; and DeMoivre’s Theorem.
- The binomial theorem for polynomial coefficients.
- Polynomials as a family of functions, especially quadratic polynomials; including the construction of sequences of polynomials with distinct roots that limit to a polynomial with repeated roots; sequences of polynomials with all real roots that limit to polynomials with complex non-real roots; and explaining these limit in terms of graphical as well as algebraic representations.
- Taylor and Maclaurin series, and limits of convergent sequences of functions.
V. Representations, Definitions, and Using Mathematical Parallels: Exponential, Logarithmic, and Linear Functions

Supporting the Work of Teaching
At the end of this module, you should be able to:

- Draw parallels, using mathematically careful language, between the definition of linear and exponential functions and their features (e.g., slope, growth rate), across graphical, tabular, verbal, and algebraic representations.
- Evaluate the affordances of alternative definitions of slope in terms of graphical, tabular, verbal, and algebraic representations.

Mathematical Knowledge
At the end of this module, you should be able to explain and use, in a mathematically careful way, with well-chosen notation and/or families of examples:

- Exponential growth and exponential functions; the analogy between linear functions and exponential functions, and addition and multiplication.
- The definition of exponents as a case of fitting cases of a definition to mathematically desired conditions.
- Logarithmic functions as inverses of exponential functions.
- Alternative definitions of $e$ and how they relate.
Course Requirements

The following is a list of course requirements. Detailed guidelines and evaluation rubrics will be available on cTools.

Participation

This class will build on your experiences as learners, observers, and doers of mathematics and mathematics teaching. Moreover, an integral part of the work of teaching is talking through and listening to mathematical ideas in a collegial, respectful, and professional manner. You will receive 2 points for your participation in each class meeting. If you must miss a class, you must contact me prior to the class.

The remaining points for participation will come from a combination of:

- Coming up to the board at least once during the term to explain a piece of mathematics in response to a question or prompt, either from a student or from me.
- At least one 20-minute meeting with me before the end of the term.
- Your analysis assignments.
- Preparedness for participation, including doing readings, looking up information, and bringing copies of your analyses and other materials as described in the course schedule.
- Your collaboration with your Final Project team members.

Analysis Assignments

A few times throughout the term, you will be asked to submit an analysis of a mathematics problem or of a set of mathematical explanations online. These are meant to prepare you for discussion, as well as inform how to guide the discussions on these topics.

Problem Sets

Most weeks, you will be assigned a few problems to write up. The quality of the work depends on mathematical soundness as well as how carefully, coherently, and completely you explain your reasoning. Being coherent and careful includes leaving out irrelevant – and therefore possibly distracting – content. Part of teaching is identifying what mathematical content is relevant for an explanation, and your problem sets are an opportunity to hone this skill.

Performance Exams

There will be two performance exams, one in the middle of the term, and one near the end of the term. We will discuss these more in class.
Written Exams

There will be a midterm exam and a final exam. The expectations for your work on these exams are similar to problem set expectations. The mathematics on these exams will address the mathematical knowledge goals in the course modules.

Final Project

A description of the final project is contained in the next section of this syllabus.

Final Project

Your project, to be completed in teams of 5-6 students, will include a paper to be posted online as well a presentation to the class. Each project will analyse one of the following:

1. Unique Prime Factorization and Euclid’s Lemma.
2. Proof by Contradiction, if and only if, and if/then statements, converse, inverse, contrapositive.
4. The long-division algorithm for polynomials as a parallel to the long-division algorithm for numbers.
5. The relationship between the long-division algorithm for polynomials, the roots of a polynomial, and factors of a polynomial, and the degree of a polynomial.
6. Function composition and function inverses, including the notion of left inverses and right inverses.
7. Shapes of polynomials and rationals (based on the precalculus textbook by Connally, Hughes-Hallett, et al.).
9. Wildcard: you propose something! If you choose this option, please write a paragraph proposal and email it to Dr. Lai and Ms. Mende.

Written Component

Your paper should include:

1. Are you wondering about the page limit? There is no strict page limit – you should write (and revise!) until you have a coherent, succinct, and thoughtful argument on your ideas for the components laid out here (make sure to follow the guidelines and the rubric), including examples that provide evidence for the claims you are making. Most papers will probably be 12-15 pages, 1.5-spaced or
• An analysis of your topic with respect to the following components of an instructional explanation:
  - Examples that illustrate key mathematical principles, examples of “legal/illegal” processes related to the topic, and another key potential error to watch for in students.
  - Description and examples of multiple representations of the concept, and how they are linked. In the case of numbers, representations should at least include ones involving numerals and the number line. In the case of functions, representations should at least include graphical, verbal, tabular, and algebraic ones.

• A proving/reasoning task related to your topic, and an analysis of how work on the task would support the Common Core Standards.
  - Your task should support at least three of the following mathematical practices identified in the Common Core Standards: #3, #5, #6, #7, #8.
  - You should state the task, provide one complete solution to the task, and sketches of alternate solution paths.
  - You should identify the conditions of the task, and explain what kind of statement is being reasoned about in terms of Hyman Bass’s proposed classification.
  - In your analysis of the task, identify the mathematical knowledge involved in enacting the task. Describe the structure of solutions to the task: how do solutions use the conditions of the problem? In your complete solution, what representations are deployed? Would the representations deployed in explaining the task statement differ from those for the solution, or in the alternate solutions? What notational considerations are there? What definitions are used? How might solutions change if alternative definitions had been used? What other language considerations are there?

This task could be one of your own making – or it could be chosen from or modified from one in a textbook or other resource. The work of a teacher includes evaluation of tasks in terms of instructional purpose; here, you are engaging in the selection of a task based upon your evaluation of its merit as a proving/reasoning task. The quality of the task you write about will be judged on its quality as a proving/reasoning task, not where it comes from.

An elaboration of the guidelines and submission process is forthcoming.

The Common Core Standards for Mathematical Practice, and Bass’s proposed classification may be found in this syllabus under “Additional References”.

Presentation Component

Each presentation will be 15-20 minutes in length. All team members should address the entire class at some point during the presentation. Your presentation to the class should include:

double-spaced (including diagrams and tables). It doesn’t matter what font or spacing you use as long as it is legible (for example, Palatino at 10 pt size, 1.5 spacing, with 1-inch margins is an example of a reasonable choice).

• A presentation of your proof/reasoning task, in which you pose either the task or a scaffolding task for the class to engage in.
• Questions to the class that assess their understanding of the presented task.
• An explanation of why you chose those discussion questions to ask, what pieces of the solution or task it highlights, add alternative pieces of the solution you ultimately decided against highlighting in this presentation.

Further guidelines for the presentation are forthcoming.

Course Evaluation

Your grade will be calculated as follows:

<table>
<thead>
<tr>
<th>Component</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class Participation</td>
<td>70</td>
</tr>
<tr>
<td>Analysis Assignments</td>
<td>(counted in participation grade)</td>
</tr>
<tr>
<td>Problem Sets</td>
<td>180</td>
</tr>
<tr>
<td>Performance Exam I</td>
<td>100</td>
</tr>
<tr>
<td>Performance Exam II</td>
<td>100</td>
</tr>
<tr>
<td>Written Midterm Exam</td>
<td>150</td>
</tr>
<tr>
<td>Written Final</td>
<td>150</td>
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<tr>
<td>Final Project</td>
<td>150</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>900</td>
</tr>
</tbody>
</table>

Late assignments will not be accepted, just as teachers are not allowed to submit grades and reports late.

Schedule for Math486-W11

A full schedule for this course may be found on the cTools site.

Notable Dates relevant to Final Projects:

Jan 26: Email Dr. Lai and Ms. Mende with your final presentation group members.
Jan 31: Email Dr. Lai and Ms. Mende with your group’s project preferences (rank the above projects in order of preference) and presentation date preference.
Feb 4: Match Day! Dr. Lai and Ms. Mende will email you your group’s assigned project and presentation date preference.
Feb 14: By this date, team members should have met with either Dr. Lai or Ms. Mende for a 1/2 hour appointment regarding your team project. At least half the group members must be present at these meetings (and if the whole team is not present, then the other team members should write an email memo to the other members regarding the content of the meeting, cc’d to Dr. Lai and Ms. Mende).
March 16: First draft of written portion of final presentation due. This draft should contain at least an outline of all planned content. The more you have, the better feedback we can give you, and the more you can benefit.
April 4: By this date, team members should have met with either Dr. Lai or Ms. Mende for a 1/2 hour appointment about the first draft. At least half the group members must be present at these meetings (and if the whole team is
not present, then the other team members should write an email memo to
the other members regarding the content of the meeting, cc’d to Dr. Lai and
Ms. Mende).
April 6, 13: Presentations! Faculty from the School of Education and the Department of
Mathematics, as well as alumni of previous Math486 classes, will be invited
to attend. Please feel free to invite your friends to join us.
April 21: Written component of Final Project due. Turn in both a paper copy in the
envelope at East Hall 1856 and a PDF copy as an email attachment to Dr. Lai
and Ms. Mende. All members of your team should be cc’d in this email.
### Types of Mathematical Claims

**Claims mathematicians prove**

Hyman Bass, distributed at the Elementary Mathematics Laboratory 2010

**What kinds of claims do mathematicians prove?**

<table>
<thead>
<tr>
<th>Type of Claim</th>
<th>Examples</th>
</tr>
</thead>
</table>
| **1. Generality**  
Something observed in some cases remains true in all cases, or in some more general setting. | a) Odd + Odd = Even. True for all integers; no longer meaningful for fractions.  
b) Equal sides implies equal angles. True for all triangles, but not for quadrilaterals. |
| **2. Properties**  
Features or properties of certain mathematical objects, possibly to characterize them uniquely. | a) Every positive integer is a sum of four squares (e.g., $5 = 1^2 + 1^2 + 1^2 + 1^2$; $9 = 1^2 + 1^2 + 2^2 + 2^2$)  
b) The decimal expansion of a fraction is eventually repeating  
c) A whole number and the sum of its digits leave the same remainder when divided by 9.  
d) The sum of the angles of a triangle is $180^\circ$  
e) Numbers $0 < a \leq b \leq c$ are the side lengths of a right triangle if and only if $a^2 + b^2 = c^2$  
f) The $3, 4, 5$ triangle is the smallest right triangle with whole number side lengths. |
| **3. Classification**  
Finding (and describing) all examples of a class of mathematical objects, for example all solutions to a problem. | a) “I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how much money might I have?”  
b) The five regular ("Platonic") solids  
c) All quadrilaterals with a line of symmetry  
d) A whole number is **perfect** if it is the sum of its proper divisors. (E.g. $6 = 1+2+3$) What are all perfect numbers? For example, are there infinitely many? |
| **4. Equivalence**  
That two or more mathematical things are the "same." | a) Finding all 3-digit numbers with digits 1, 2, 3 is the same as finding all 3-car trains made with a green, a purple, and a yellow car.  
b) $\frac{3 \cdot 15}{4 \cdot 20} = 0.75$  
c) $1+3+5 + \ldots + (2n-1) = n^2$  
(The sum of the first $n$ odd numbers is $n^2$) |
| **5. Existential**  
That something (like a solution to a problem) exists or does not exist. | a) There is no solution to $7 \div 0 = x$  
b) $\sqrt{2}$ is not a fraction  
c) If $n \geq 3$, the equation $x^n + y^n = z^n$ has no solution with positive whole numbers $x, y, z$. |

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*The different types listed below are not disjoint. Some claims fit more than one category*
Overview

http://www.corestandards.org/the-standards/mathematics

These Standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the students mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as \((a + b)(x + y)\) and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding \((a + b + c)(x + y)\). Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

The Standards set grade-specific standards but do not define the intervention methods or materials necessary to support students who are well below or well above grade-level expectations. It is also beyond the scope of the Standards to define the full range of supports appropriate for English language learners and for students with special needs. At the same time, all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives. The Standards should be read as allowing for the widest possible range of students to participate fully from the outset, along with appropriate accommodations to ensure maximum participation of students with special education needs. For example, for students with disabilities reading should allow for use of Braille, screen reader technology, or other assistive devices, while writing should include the use of a scribe, computer, or speech-to-text technology. In a similar vein, speaking and listening should be interpreted broadly to include sign language. No set of grade-specific standards can fully reflect the great variety in abilities, needs, learning rates, and achievement levels of students in any given classroom. However, the Standards do provide clear signposts along the way to the goal of college and career readiness for all students.

Standards for Mathematical Practice


The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Councils report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and ones own efficacy).
1. **Make sense of problems and persevere in solving them.** Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, Common Core State Standards for Mathematics standards for mathematical practice communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and if there is a flaw in an argument explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
4. Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision. Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see $7 \times 8$ equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$,
older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$.

8. Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content
The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.
Michigan Merit Curriculum Content Expectations (Excerpts)

Precalculus Content Expectations, Standards P1, P2, P4, P6:
Functions and Various Families of Functions

P1 Functions

P1.1 Know and use a definition of a function to decide if a given relation is a function.

P1.4 Determine whether a function (given symbolically or graphically) has an inverse and express the inverse (symbolically, if the function is given symbolically, or graphically, if given graphically) if it exists. Know and interpret the function notation for inverses.

P1.5 Determine whether two given functions are inverses, using composition.

P1.7 Understand the concept of limit of a function as \( x \) approaches a number or infinity.

P1.8 Explain how the rates of change of functions in different families (e.g., linear functions, exponential functions, etc.) differ, referring to graphical representations.

P2 Exponential and Logarithmic Functions

P2.1 Use the inverse relationship between exponential and logarithmic functions to solve equations and problems.

P2.5 Explain how the parameters of an exponential or logarithmic model relate to the data set or situation being modeled. Find an exponential or logarithmic function to model a given data set or situation. Solve problems involving exponential growth and decay.

P4 Polynomial Functions

P4.1 Given a polynomial function whose roots are known or can be calculated, find the intervals on which the functions values are positive and those where it is negative.

P4.2 Solve polynomial equations and inequalities of degree greater than or equal to three. Graph polynomial functions given in factored form using zeros and their multiplicities, testing the sign-on intervals and analyzing the functions large-scale behavior.

P4.3 Know and apply fundamental facts about polynomials: the Remainder Theorem, the Factor Theorem, and the Fundamental Theorem of Algebra.

P6 Trigonometric Functions

P6.1 Define (using the unit circle), graph, and use all trigonometric functions of any angle. Convert between radian and degree measure. Calculate arc lengths in given circles.

P6.3 Know basic properties of the inverse trigonometric functions \( \sin^{-1}x \), \( \cos^{-1}x \), \( \tan^{-1}x \), including their domains and ranges. Recognize their graphs.

Geometry Content Expectations, Standard L3:
Mathematical Reasoning, Logic, and Proof

L3.1 Mathematical Reasoning

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L3.1.3 Define and explain the roles of axioms (postulates), definitions, theorems, counterexamples, and proofs in the logical structure of mathematics. Identify and give examples of each.

L3.2 Language and Laws of Logic

L3.2.1 Know and use the terms of basic logic.
L3.2.2 Use the connectives “not,” “and,” “or,” and “if..., then,” in mathematical and everyday settings. Know the truth table of each connective and how to logically negate statements involving these connectives.
L3.2.4 Write the converse, inverse, and contrapositive of an “If..., then...” statement. Use the fact, in mathematical and everyday settings, that the contrapositive is logically equivalent to the original while the inverse and converse are not.

L3.3 Proof

L3.3.1 Know the basic structure for the proof of an “If..., then...” statement (assuming the hypothesis and ending with the conclusion) and that proving the contrapositive is equivalent.
L3.3.2 Construct proofs by contradiction. Use counterexamples, when appropriate, to disprove a statement.
L3.3.3 Explain the difference between a necessary and a sufficient condition within the statement of a theorem. Determine the correct conclusions based on interpreting a theorem in which necessary or sufficient conditions in the theorem or hypothesis are satisfied.