Fundamental Theorem of Algebra

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We prove the Fundamental Theorem of Algebra:

Fundamental Theorem of Algebra. *Let f be a non-constant polynomial with real coefficients. Then f has at least one complex root.*

The first half of the proof analyses the images of "sufficiently large" loops (Section 1) and "sufficiently small" loops (Section 2). We will quantify in these sections what "sufficiently large" and "sufficiently small" mean. The second half of the proof analyses the images of loops "close together" (Section 3), and applies the properties of images to show the existence of a $z \in \mathbb{C}$ so that $f(z) = 0$ (Section 4). The existence proof relies on an examination of winding number.

0 Preliminaries

Let f be defined with the coefficients

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n,$$

where $a_n \neq 0$ (to guarantee that $f(z)$ is nonconstant).

Note that since f is nonconstant, $n \geq 1$.

Notation. Throughout the proof, we will use the notation:

- $a = \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|, 1\}$ coefficients of f.
- $C_R = (R, t), t \in [0, 2\pi]$, where (R, t) is the parametrization, written in polar form, of the counterclockwise path in $\mathbb{C}$ traversing once counterclockwise around the circle with radius R and center 0.

Simplifying Assumptions. Note that if $a_0 = 0$, then $f(0) = 0$, so 0 is a root. So from now on, we work with the case $a_0 \neq 0$.

It is sufficient to prove the theorem for $a_n = 1$. For suppose that we have shown that all nonconstant polynomials with leading coefficient 1 must have at least one complex root. Then the polynomial

$$g(z) = \frac{f(z)}{a_n} = \frac{a_0}{a_n} + \frac{a_1}{a_n} z + \ldots + \frac{a_{n-1}}{a_n} z^{n-1} + z^n$$

has at least one root; suppose the root is z_0. It follows that

$$f(z_0) = a_n g(z_0) = a_n \cdot 0 = 0,$$

so z_0 is a root of f. Thus proving the Fundamental Theorem of Algebra for the case $a_n = 1$ implies the theorem for the case when $a_n \neq 1$. So, in what follows, we assume $a_n = 1$.

1 Large Input Loops

Our goal in this section is to prove Proposition 1 below. The key idea of its statement is –

If R is big enough, then f sends C_R to a path contained in an annulus around the circle with radius R^n, centered around the origin.

A schematic of the statement can be found at the end of this section.

Proposition 1. *If $R \geq 2na$, then*

$$f(C_R) \subset \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{2}R^n \leq |z| \leq \frac{3}{2}R^n \right\}.$$

Lemma 1. *If $R \geq 2na$, then*

$$aR^k \leq \frac{R^n}{2n}$$

for $0 \leq k \leq n-1$.

Private math: For this lemma, we seek an R so that $aR^k \leq \frac{R^n}{2n}$, which can be rewritten

$$\begin{aligned} aR^k &\leq \frac{R^n}{2n} \\ 2na &\leq R^{n-k} \\ \sqrt[n-k]{2na} &\leq R \quad (*) \end{aligned}$$

One R that satisfies $(*)$ is $2na$, because $2na$ is at least as large as $\sqrt[n-k]{2na}$.

Proof of Lemma 1. Note that $2na > 1$, so if $R \geq 2na$, then $R^m \geq R \geq 2na$ for $m \geq 1$, including $m = n - k$, for $0 \leq k \leq n-1$. It follows that

$$\begin{aligned} R^{n-k} &\geq 2na \\ R^n &\geq 2naR^k \\ \frac{R^n}{2n} &\geq aR^k \end{aligned}$$

as desired. □

Proof of Proposition 1. To prove the proposition, we need to show that $R \geq 2na$ implies the radius of $f(C_R(t))$ is between $\frac{1}{2}R^n$ and $\frac{3}{2}R^n$, for all t. The radius of $f(C_R(t))$ is

$$\left|(R,t)^n + a_{n-1}(R,t)^{n-1} + \ldots + a_1(R,t) + a_0\right|,$$

which is bounded below by

$$\left| R^n - \left(|a_{n-1}|R^{n-1} + \ldots + |a_1|R + |a_0|\right) \right|, \quad (*)$$

and bounded above by

$$R^n + |a_{n-1}|R^{n-1} + \ldots + |a_1|R + |a_0|. \quad (**)$$

Suppose $R \geq 2na$. We have just shown that this condition implies $aR^k \leq \frac{R^n}{2n}$ for $0 \leq k \leq n-1$. Consequently,

$$|a_{n-1}R^{n-1}| + \ldots + |a_1R| + |a_0| \leq \underbrace{\frac{R^n}{2n} + \ldots + \frac{R^n}{2n} + \frac{R^n}{2n}}_{n \text{ times}} = \frac{R^n}{2}.$$

Plugging in our calculations into $(*)$ and $(**)$, we conclude that $R \geq 2na$ implies the radius of $f(C_R(t))$ satisfies

$$\left| R^n - \frac{R^n}{2} \right| \leq |f(C_R(t))| \leq R^n + \frac{R^n}{2}$$

$$\frac{1}{2}R^n \leq |f(C_R(t))| \leq \frac{3}{2}R^n$$

for all times t, as claimed. □

We have shown that when R is "large enough", meaning its radius is at least $2na$, then f sends C_R to a loop that is contained in a neighbourhood of a circle with radius R^n. See Figure 1 for a visual schematic of the meaning of Proposition 1.

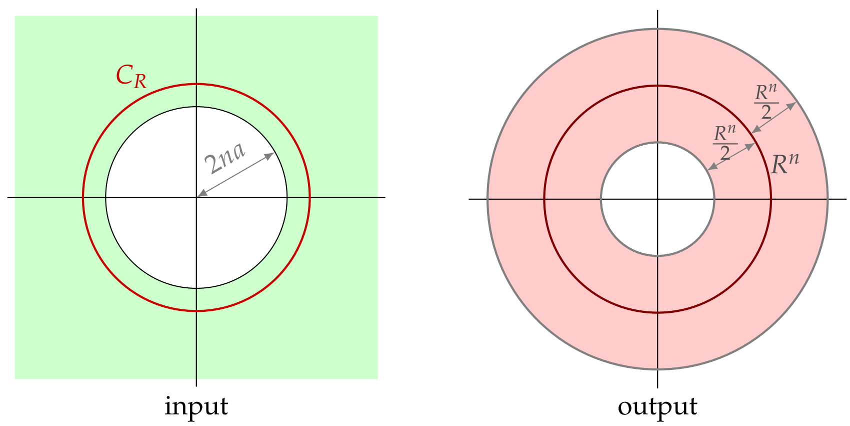

Figure 1. Visual schematic for Proposition 1. If C_R is contained in the shaded input region, then its image is contained in the shaded output region.

2 Small Input Loops

Proposition 2. *When $R \leq \frac{|a_0|}{2na}$, then $f(C_R)$ is contained in a disk of radius $\frac{|a_0|}{2}$ centered at a_0.*

Lemma 2. *Suppose $R \leq \frac{|a_0|}{2na}$. Then $aR^k \leq \frac{|a_0|}{2n}$ for $1 \leq k \leq n$.*

Private math: For this lemma, we seek an R so that $aR^k \leq \frac{|a_0|}{2n}$, which can be rewritten

$$\begin{aligned} aR^k &\leq \frac{|a_0|}{2n} \\ R^k &\leq \frac{|a_0|}{2na} \\ R &\leq \sqrt[k]{\frac{|a_0|}{2na}} \quad (*) \end{aligned}$$

One R that satisfies $(*)$ is $\frac{|a_0|}{2na}$, because $\frac{|a_0|}{2na}$ is at most $\sqrt[k]{\frac{|a_0|}{2na}}$ if $\frac{|a_0|}{2na} < 1$.

Proof of Lemma 2. Suppose $R \leq \frac{|a_0|}{2na}$, and let $1 \leq k$. Note that $\frac{|a_0|}{2na} < 1$, as $|a_0|$ is at most a. This implies $\left(\frac{|a_0|}{2na}\right)^k \leq \frac{|a_0|}{2na}$.

We have

$$aR^k \leq a \cdot \left(\frac{|a_0|}{2na}\right)^k \leq a \cdot \frac{|a_0|}{2na} = \frac{|a_0|}{2n}$$

as desired. □

Proof of Proposition 2. To prove the proposition, we need to show that $R \leq \frac{|a_0|}{2na}$ implies that $|a_0 - f(C_R(t))| \leq \frac{|a_0|}{2}$ for all t. Note that

$$\begin{aligned} |a_0 - f(C_R(t))| &= \left|a_0 - \left(a_0 + a_1(R,t) + \ldots + a_{n-1}(R,t)^{n-1} + (R,t)^n\right)\right| \\ &\leq |a_1|R + |a_2|R^2 + \ldots + |a_{n-1}|R^{n-1} + R^n. \end{aligned}$$

Suppose $R \leq \frac{|a_0|}{2na}$. We have just shown this condition implies $aR^k \leq \frac{|a_0|}{2n}$ for $1 \leq k$. Consequently,

$$\begin{aligned} |a_0 - f(C_R(t))| &\leq |a_1|R + |a_2|R^2 + \ldots + |a_{n-1}|R^{n-1} + R^n \\ &\leq \underbrace{\frac{|a_0|}{2n} + \frac{|a_0|}{2n} + \ldots + \frac{|a_0|}{2n}}_{n \text{ times}} = \frac{|a_0|}{2} \end{aligned}$$

for all t, as claimed. □

We have shown that when C_R is "small enough", meaning its radius is at most $\frac{|a_0|}{2na}$, then f sends C_R to a loop that is contained in a neighbourhood of a_0, the constant term.

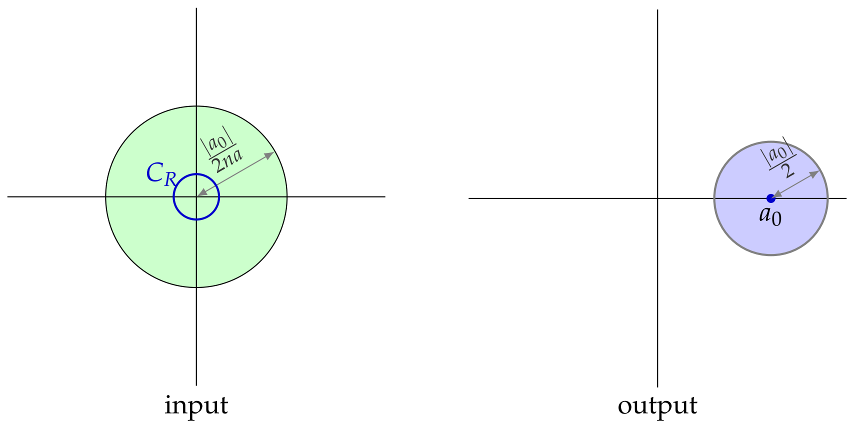

Figure 2. Visual schematic for Proposition 2. If C_R is contained in the shaded input region, then its image is contained in the shaded output region.

3 Neighbourhoods of Input Loops

The goal of this section is to prove Proposition 3.

Proposition 3. *Let* $(R, t) \in \mathbb{C}$. *For every* $\epsilon > 0$, *there exists a* $\delta_0 > 0$ *such that* $|\delta| < \delta_0$ *implies*

$$|f(R+\delta, t) - f(R, t)| < \epsilon.$$

Recall that the degree of f is n. In the proof of Proposition 3, we let C denote the largest coefficient over all terms of the multinomials $\{(x+\delta)^k \mid 0 < k \leq n\}$.

Lemma 3. *Given* $0 < |\delta| < R$,

$$|(R+\delta)^k - R^k| \leq |\delta| k C R^{k-1}$$

for all integers $k \geq 0$.

Proof of Lemma 3. After expansion, the expression $(R+\delta)^k - R^k$ contains only terms of the form $R^{k-1}\delta$, $R^{k-2}\delta^2, \ldots, R\delta^{k-1}$, δ^k, as the term R^k cancels.

To prove the lemma, first consider positive δ. Then

$$\begin{aligned}
|(R+\delta)^k - R^k| &= (R+\delta)^k - R^k \\
&\leq CR^{k-1}\delta + CR^{k-2}\delta^2 + \ldots + CR\delta^{k-1} + C\delta^k \\
&\leq \delta C(R^{k-1} + R^{k-2}\delta + \ldots + R\delta^{k-2} + \delta^{k-1}) \\
&\leq \delta C(\underbrace{R^{k-1} + R^{k-1} + \ldots + R^{k-1}}_{k \text{ times}}) \\
&\leq \delta C k R^{k-1}.
\end{aligned}$$

Now suppose δ is negative, so $|(R+\delta)^k - R^k| = R^k - (R-|\delta|)^k$. Because power functions are concave up, it follows that

$$R^k - (R-|\delta|)^k \leq (R+|\delta|)^k - R^k \leq |\delta| CkR^{k-1}.$$

We have shown the lemma. □

Corollary 1. *Let $K = an^2 C \max\{R,1\}^n$. If $0 \leq |\delta| \leq R$, then*

$$|f(R+\delta,t) - f(R,t)| < \delta K.$$

Proof of Corollary 1. Note that

$$\begin{aligned}
|f(R+\delta,t) - f(R,t)| &= \left| \sum_{k=1}^{n} a_k \left((R+\delta,t)^k - (R,t)^k \right) \right| \\
&\leq \sum_{k=1}^{n} a \left((R+\delta,t)^k - (R,t)^k \right) \\
&\leq \sum_{k=1}^{n} a \left((R+\delta)^k - R^k \right).
\end{aligned}$$

It follows from Lemma 3 that

$$\begin{aligned}
\sum_{k=1}^{n} a \left((R+\delta)^k - R^k \right) &\leq \sum_{k=1}^{n} a\delta kCR^{k-1} \\
&< \sum_{k=1}^{n} a\delta nC \max\{R,1\}^n \\
&= n \cdot a\delta nC \max\{R,1\}^n \\
&= \delta K
\end{aligned}$$

Hence $|f(R+\delta,t) - f(R,t)| < \delta K$ as claimed. □

We now prove the proposition.

Proof of Proposition 3. Fix $\epsilon > 0$. We seek $\delta_0 > 0$ that guarantees that if $|\delta| < \delta_0$, then $|f(R+\delta,t) - f(R,t)| < \epsilon$. We claim that

$$\delta_0 = \frac{\epsilon}{K}$$

works. Indeed, suppose

$$|\delta| < \delta_0 = \frac{\epsilon}{K}.$$

By Corollary 1,

$$|f(R+\delta,t) - f(R,t)| < |\delta| K < \delta_0 K = \frac{\epsilon}{K} \cdot K = \epsilon$$

as we sought to prove. □

4 Proof of the Fundamental Theorem of Algebra

Proposition 4. *Let $R > 0$, and suppose $f(C_R)$ does not cross the origin. Let ϵ_0 be* $\min\{|f(C_R(t))| \mid t \in [0, 2\pi]\}$. *Let $\delta_0 = \epsilon_0/K$, where K is defined as in Corollary 1. Then $|\delta| < \delta_0$ implies that the winding numbers of $f(C_R)$ and $f(C_{R+\delta})$ are the same.*

Proof. We show that there is a deformation of $f(C_R)$ to $f(C_{R+\delta})$ that does not cross the origin, and hence the winding numbers of the two paths must be equal.

Define $p_\alpha(t), t \in [0, 2\pi]$ as the path $f(C_R(t)) + \alpha(f(C_{R+\delta}(t)) - f(C_R(t)))$ for $\alpha \in [0, 1]$. Note that $p_0 = f(C_R)$ and $p_1 = f(C_{R+\delta})$, and as α sweeps from 0 to 1, the path p_α deforms from $f(C_R)$ to $f(C_{R+\delta})$.

By Proposition 3, $|\delta| < \delta_0$ guarantees that $|f(C_{R+\delta}(t)) - f(C_R(t))|$ is strictly less than ϵ_0, the minimum radius of any point in $f(C_R(t))$. It follows that p_α cannot pass through the origin for any α. Hence we can deform $f(C_R)$ to $f(C_{R+\delta})$ without passing through the origin, so $|\delta| < \delta_0$ guarantees that the winding numbers of $f(C_R)$ and $f(C_{R+\delta})$ are equal. □

By using similar techniques, we obtain the corollary of Proposition 1:

Corollary 2. *Suppose $R \geq 2na$. Then the winding number of $f(C_R)$ is n.*

Proof. We show that the winding number of $f(C_R)$ is n by deforming a path that has winding number n to $f(C_R)$. This path is $(C_R)^n = (R^n, nt)$.

Define the deformation $q_\alpha(t) = (C_R)^n + \alpha(f(C_R(t)) - (C_R(t))^n), t \in [0, 2\pi]$ from $(C_R)^n$ to $f(C_R)$. By Proposition 1, $|f(C_R(t))|$ is at most $R^n/2$. It follows that $|f(C_R(t)) - (C_R(t))^n| \leq |R^n/2 - R^n| = R^n/2$, so q_α does not pass through the origin for any α. Thus the winding numbers of $(C_R)^n$ and $f(C_R)$ must be equal. □

We now complete the proof of the Fundamental Theorem of Algebra. In the following proof, we use *circle* to denote a path of the form $(r,t), t \in [0, 2\pi]$, where $r > 0$ is constant.

Fundamental Theorem of Algebra. *Let f be a nonconstant polynomial. Then f has at least one complex root.*

Proof of Fundamental Theorem of Algebra. In this argument, we analyse winding numbers. Recall $w(p)$ denotes the winding number of p.

Suppose by way of contradiction that f has no roots. Then, for any $R > 0$, the path $f(C_R)$ does not cross the origin. In particular, $w(f(C_R))$ is well-defined. By Proposition 4, there is an annulus $N_\delta(C_R)$ such that if C is a circle in $N_\delta(C_R)$, then $w(f(C)) = w(f(C_R))$. For each R, pick such an annulus and denote it $N(C_R)$.

Consider the region $D = \{z \in \mathbb{C} \mid 0 < |z| \leq 2na\}$. This region is covered by the collection $\{N(C_R) \mid 0 < R \leq 2na\}$ of annuli. Circles from overlapping annuli must produce images with the equal winding numbers. Because the region D is connected and covered by annuli, each of which overlaps at least one other annulus, we can conclude that the winding number of the images of all circles in D must be the same.

The winding number of $f(C_{2na})$ is n, by Corollary 2. Since $C_{2na} \subset D$, it follows that the winding number of images of all circles in D must be n. But by Proposition 2, the winding number of $f(C_{|a_0|/2na})$ is 0, and $0 \neq n$. Hence we have found our contradiction. There must exist a root of f in the complex plane. $\square$