A probabilistic version of the game of zombies and survivors on graphs

Xavier Pérez-Giménez†
joint work with
Anthony Bonato†, Dieter Mitsche* and Paweł Prałat†

†Ryerson University

*Université de Nice Sophia-Antipolis

Graphs @ Ryerson, September 2015
Zombie Policeman
Zombies and survivor: who wants to live forever?

**Game rules**

- 2 players (1 survivor vs. \(k\) zombies) on vertices of \(G\).
- Initial position: zombies (random); survivor (deterministic).
- At each step:
  - Zombies move first toward the survivor (ties solved at random).
  - Survivor moves next to any neighbour or stays put.
- Zombies win if one zombie eventually eats the survivor.
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Basic definitions

Zombie number

\[ z(G) = \min \left\{ k \in \mathbb{N} : P(k \text{ zombies win}) \geq 1/2 \right\} \]
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\[ z(G) = \min \left\{ k \in \mathbb{N} : \Pr(k \text{ zombies win}) \geq 1/2 \right\} \]

Observe: \( z(G) \geq c(G) \), where \( c(G) \) is the cop number.
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\[ Z(G) = \frac{z(G)}{c(G)} \]
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Theorem
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Projective plane $P_q$ of order $q$ (q prime power)

Graph $G_q$

Incidence graph of $P_q$: 
Projective plane

Projective plane $P_q$ of order $q$ ($q$ prime power)

Graph $G_q$

Incidence graph of $P_q$:
- $(P, L)$-bipartite
Projective plane \( P_q \) of order \( q \) (\( q \) prime power)

**Graph \( G_q \)**

Incidence graph of \( P_q \):
- \( (P, L) \)-bipartite
- \(|P| = |L| = q^2 + q + 1\)
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**Graph $G_q$**

Incidence graph of $P_q$:
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Projective plane $P_q$ of order $q$ (q prime power)

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\[ z(G_q) = 2q + \Theta(\sqrt{q}). \quad Z(G_q) \sim 2. \]
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Initially: \( k_P \) zombies in \( P \) and \( k_L \) zombies in \( L \) \( (k = k_P + k_L) \)
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Lemma
- \( k \leq 2q - \omega\sqrt{q} \implies \text{a.a.s.} \quad k_P, k_L \leq q - 1 \)
  (survivor strategy).
Projective plane

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**Lemma**

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- \( k \geq 2q + \omega \sqrt{n} \implies \text{a.a.s. } k_P, k_L \geq q \)  
  (zombie strategy).
Projective plane: observation
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Survivor cannot stop!
Projective plane: zombies’ strategy

It’s zombies’ turn to move…

$k_R \geq q$

$\text{deg } q+1$

$k_L \geq q$
It’s zombies’ turn to move…

$P$  \hspace{2cm} $L$

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Projective plane: zombies’ strategy

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Zombies and survivors on graphs

G@R 2015
It's zombies' turn to move...

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Projective plane: zombies’ strategy

It’s zombies’ turn to move…

\[ q + 1 \text{deg} \]

\[ k_L \geq q \]

\[ k_R \geq q \]

Zombies block all ways of escape with positive probability!

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G@R 2015 9 / 19
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It’s zombies’ turn to move...

Zombies block all ways of escape with positive probability!

\[ P \text{ deg } q+1 \Rightarrow k_R \geq q \]

\[ L \quad k_L \geq q \]
It’s the survivor’s turn to move…

\[ P \quad \text{deg} \quad q+1 \]

\[ k_r < q \]

\[ L \]

\[ k_L < q \]
Projective plane: survivor’s strategy

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$P$

$deg \ q + 1$

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$k_L < q$

The survivor can always escape for one more step!
It’s the survivor’s turn to move...

\[ P \]

\[ L \]

\[ k_r < q \]

\[ k_L < q \]

\[ \text{deg } q+1 \]

The survivor can always escape for one more step!
Projective plane: survivor’s strategy

It’s the survivor’s turn to move...

The survivor can always escape for one more step!
Vertices of $Q_n$ are $\{0, 1\}$-strings of length $n$.

$Q_n$ is $(E, O)$-bipartite

$E$ strings with even number of 1’s.

$O$ strings with odd number of 1’s.
Theorem

\[ z(Q_n) = \frac{2n}{3} + \Theta(\sqrt{n}). \quad Z(Q_n) \sim \frac{4}{3}. \]
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- \( k \leq \frac{2n}{3} - \omega \sqrt{n} \implies \text{a.a.s.} \quad k_E, k_O < \frac{n}{3} \)

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Hypercube

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  (zombie strategy).
Hypercube: survivor’s strategy

(The survivor can always find a safe start.)

$S_i =$ set of zombies at distance $i$ from survivor ($|S_1|, |S_2| < n/3$)
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At each step:

- If \( S_1 = \emptyset \), then survivor stays put.
- Otherwise, survivor can find a move away from \( S_1, S_2 \).

\[
\begin{array}{c}
E \\
s_{\text{deg } n}
\end{array}
\begin{array}{c}
O \\
S_1 \\
S_2
\end{array}
\]
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Hypercube: zombies’ strategy

Vector of distances $\vec{d} = (d_1, \ldots, d_k)$

It never increases (after each zombie move).
Hypercube: zombies’ strategy

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Uniform coordinates (shared by all players)

Every $n$ steps, $\vec{d}$ decreases with positive probability.
Hypercube: zombies’ strategy

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Uniform coordinates (shared by all players)

- If the survivor flips a non-uniform coordinate, then $\vec{d}$ decreases

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Grids

$G_n$

$n \times n$ square grid

$T_n$

$n \times n$ toroidal grid
Grids

Theorem
For $n \geq 2$, $z(G_n) = 2$. Hence, $Z(G_n) = 1$.

However...
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However...

Theorem

\( z(T_n) \geq \sqrt{n}/(\omega \log n) \), while \( c(T_n) = 3 \).
So \( Z(T_n) \geq \sqrt{n}/(\omega \log n) \).
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**Goal of the survivor:**

![Diagram of a grid with zombies and survivors]
Grids

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**Goal of the survivor:**
B \log n
A \log n

\[ \text{Torus: survivor's strategy} \]
Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
Torus: survivor’s strategy

\[ A \log n \quad \text{and} \quad B \log n \]
Torus: survivor’s strategy

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Zombies and survivors on graphs

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Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
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\[ A \log n \quad \text{and} \quad B \log n \]
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Torus: survivor’s strategy

$A \log n$  $B \log n$

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Zombies and survivors on graphs  G@R 2015  17 / 19
Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
Torus: survivor’s strategy

\[ A \log n \quad B \log n \]
Torus: survivor’s strategy

\[ A \log n \quad B \log n \]
Torus: survivor’s strategy

$A \log n$

$B \log n$
Torus: survivor’s strategy

\[ \log n \]

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Torus: survivor’s strategy

\[ A \log n \quad \text{versus} \quad B \log n \]
Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
Torus: survivor’s strategy

\[ A\log n \]

\[ B\log n \]
Torus: survivor’s strategy

A \log n

B \log n
Torus: survivor’s strategy

$A \log n$

$B \log n$
Torus: survivor’s strategy

\[ \log A \quad \log B \]

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Torus: survivor’s strategy

\[ A \log n \quad \text{and} \quad B \log n \]
Torus: survivor’s strategy

\[ A \log n \]

\[ B \log n \]
Open questions

- Upper bound on $z(T_n)$.
- Mixed cop-zombie model: how many cops are needed to lead a team of zombies?
Thank you