FOURTH VERONESE EMBEDDINGS OF PROJECTIVE SPACES

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Abstract. We prove that fourth Veronese embeddings of projective spaces satisfies property $N_9$. This settle the Ottaviani-Paoletti conjecture for fourth Veronese embeddings.

1. Introduction

Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic 0. Let $\mathcal{L}$ be a very ample line bundle on $X$. Thus $\mathcal{L}$ defines an embedding

$$X \subseteq \mathbb{P}^r = \mathbb{P}^H^0(X, \mathcal{L})$$

where $r = r(\mathcal{L}) = h^0(X, \mathcal{L}) - 1$. Let $S = \text{Sym} H^0(X, \mathcal{L})$ be the homogeneous coordinate ring of $\mathbb{P}^r$ and let $R = R(\mathcal{L}) = \oplus H^0(X, \mathcal{L}^k)$ be the homogeneous coordinate ring of $X$ embedded by $\mathcal{L}$, viewed as an $S$-module. Since the influential papers by Green [11], [12], the study of syzygies of $R$ as $S$-module have been extensively carried out. Green’s idea was that the higher the degree of $\mathcal{L}$, the simpler are the syzygies of $R$ (at least from the beginning of the resolution). When $X$ is a curve, this philosophy has proved to govern the shape of the resolution of high degree embeddings of $X$. Green proved that, when $\deg \mathcal{L} = 2g + 1 + p$, where $g$ is the genus of $X$, then the embedding of $X$ defined by $\mathcal{L}$ satisfies property $N_p$. That is $R$ is normally generated, and all syzygies up to homological degree $p$ are linear.

Following Green and Lazarsfeld [13], we define:

**Definition 1.** The Green-Lazarsfeld index of a very ample line bundle $\mathcal{L}$ on a smooth projective variety $X$ is the largest integer $p$ such that the embedding of $X$ by $\mathcal{L}$ satisfies property $N_p$. We denote it by $p(X, \mathcal{L})$.

When $\dim X \geq 2$, much less is known about the syzygies of embeddings of $X$, even in the simplest case when $X = \mathbb{P}^n$. Green proved that when $L = \mathcal{O}(d)$, then $R$ satisfies property $N_d$. Nevertheless, this is far from the actual Green-Lazarsfeld index of the Veronese embeddings as conjectured in:

**Conjecture 2** (Ottaviani-Paoletti). The $d$th Veronese embeddings of projective spaces satisfy property $N_{3d-3}$.

For simplicity, for each $p, q$ let $K_{p,q}(X, \mathcal{L})$ be the space of $p$th syzygies of $R$ of degree $p + q$. Ottaviani and Paoletti in [20] showed that $K_{3d-2,2}(\mathbb{P}^n, \mathcal{O}(d)) \neq 0$ when $n \geq 2$ and $d \geq 3$. In other words, the conjecture is sharp.

The Ottaviani-Paoletti conjecture is known for $d = 2$ by the work of Józefiak-Pragacz-Weyman [15] and also known for $\mathbb{P}^1$ and $\mathbb{P}^2$ by the work of Green [11]
and Birkenhake [3]. See also the work of Gallego and Purnaprajna [10] for more
general result about rational surfaces. In [25], we proved the conjecture for the case
of third Veronese embeddings. For some other partial results, Rubei [22] proved
that the third Veronese embeddings of projective spaces satisfy property $N_4$. This
was generalized by Bruns, Conca and Römer [6] where they showed that the $d$th
Veronese embeddings of projective spaces satisfy property $N_{d+1}$. In a preparation
work [26], we improve that to $N_{2d-2}$.

When $d = 4$, in this paper, we prove

Main Theorem. The fourth Veronese embeddings of projective spaces satisfy prop-
erty $N_9$.

Since the Green-Lazarsfeld index of Veronese embeddings have been determined
for $\mathbb{P}^2$, we may assume that $n \geq 3$. Since we know more about the syzygies
of embeddings of curves, we could try to take general hyperplane sections of the
embeddings of $\mathbb{P}^n$ to bring it to the case of curves. Nevertheless, as we will see
in section 2, most of the time we get into trouble as the curve obtained from the
process would have the degree of the embedding less than the degree of its canonical
divisor. Moreover, the curve would lie on a surface of general type, over which our
knowledge of their syzygies are very limited. We will see that in the case of fourth
Veronese embedding of $\mathbb{P}^3$, taking general hyperplane sections, we get a canonical
curve lying on a K3 surface. Recently, Green’s conjecture has been proved in this
case by Aprodu and Farkas [1]. This reduces the determination of $p(\mathbb{P}^3, \mathcal{O}(4))$ to the
computation of the Clifford index of the curve which is the complete intersection
of two quartic surfaces in $\mathbb{P}^4$. We will define the Clifford index and compute it in
our situation in section 2.

We will now introduce a different approach to Ottaviani-Paoletti conjecture using
representation theory and combinatorics. Note that representation theory comes
into play naturally in the land of syzygies of Segre-Veronese varieties. This fact
has been exploited and successfully used in certain problems, for example see [15],
[18], [21]. We will now switch the notation a little bit to move to the world of
representation theory of general linear groups. For unexplained terminology, we
refer to the book by Fulton and Harris [9].

Let $k$ be a field of characteristic 0. Let $V$ be a finite dimensional vector space
over $k$ of dimension $n + 1$. The projective space $\mathbb{P}(V)$ has coordinate ring naturally
isomorphic to $\text{Sym} V$. For each natural number $d$, the $d$-th Veronese embedding
of $\mathbb{P}(V)$, which is naturally embedded into the projective space $\mathbb{P}(\text{Sym}^d V)$ has
coordinate ring $\text{Ver}(V, d) = \bigoplus_{k=0}^{\infty} \text{Sym}^{kd} V$. For each set of integers $p, q, b$, let
$K_{p,q}^d(V, b)$ be the associated Koszul cohomology group defined as the homology of the
3-term complex

$$
\bigwedge^{p+1} \text{Sym}^d V \otimes \text{Sym}^{(q-1)d+b} V \rightarrow \bigwedge^p \text{Sym}^d V \otimes \text{Sym}^{qd+b} V \\
\rightarrow \bigwedge^{p-1} \text{Sym}^d V \otimes \text{Sym}^{(q+1)d+b} V.
$$

Then $K_{p,q}^d(V, b)$ is the space of minimal $p$-th syzygies of degree $p + q$ of the $\text{GL}(V)$-
module $\text{Ver}(V, d, b) = \bigoplus_{k=0}^{\infty} \text{Sym}^{kd+b} V$. We write $K_{p,q}^d(b) : \text{Vect} \rightarrow \text{Vect}$ for the
functor on finite dimensional $k$-vector spaces that assigns to a vector space $V$ the
corresponding syzygy module \( K_{p,q}^d(V,b) \). In this notation, the Ottaviani-Paoletti conjecture is:

\[(1.1) \quad K_{p,q}^d(V,0) = 0 \text{ for } q \geq 2 \text{ and } p \leq 3d - 3.\]

Moreover, the Veronese modules \( \text{Ver}(V,d,b) \) are Cohen-Macaulay, the equation (1.1) can be replaced by

\[(1.2) \quad K_{p,2}^d(V,0) = 0 \text{ for } p \leq 3d - 3.\]

From the definition, it is clear that \( K_{p,q}^d(V,b) \) are \( \text{GL}(V) \)-representations, in particular, the functors \( K_{p,q}^d(b) \) are polynomial functors and decompose into irreducible polynomial functors, i.e. Schur functors. By a result of Karaguerian, Reiner and Wachs [16], these decompositions are closely related to decompositions of homology groups of matching complexes into irreducible representations as representations of symmetric groups.

**Definition 3 (Matching Complexes).** Let \( d > 1 \) be a positive integer and \( A \) a finite set. The matching complex \( C_A^d \) is the simplicial complex whose vertices are all the \( d \)-element subsets of \( A \) and whose faces are \( \{ A_1, \ldots, A_r \} \) so that \( A_1, \ldots, A_r \) are mutually disjoint.

The symmetric group \( S_A \) acts on \( C_A^d \) by permuting the elements of \( A \) making the homology groups of \( C_A^d \) representations of \( S_A \). For each partition \( \lambda \), we denote by \( V^\lambda \) the irreducible representation of \( S_{|\lambda|} \) corresponding to the partition \( \lambda \), and \( S_\lambda \) the Schur functor corresponding to the partition \( \lambda \). For each vector space \( V \), \( S_\lambda(V) \) is an irreducible representation of \( \text{GL}(V) \). The result of Karaguezian, Reiner and Wachs implies that the equation (1.2), and so the Ottaviani-Paoletti conjecture is equivalent to:

**Conjecture 4.** The only non-zero homology groups of \( C_{kd}^d \) for \( k = 1, \ldots, 3d - 1 \) is \( \tilde{H}_k - 2 \).

Besides the connection to the syzygies of Veronese embeddings, the study of connectivity of matching complexes is also of interest among the combinatorialists [4], and have connection to problems in group theory (see [17] and [23] and references therein).

For simplicity, we assume \( d = 4 \) throughout the rest of the introduction. In this case, for each \( n \geq 9 \), we have the following equivariant long exact sequence originated from Bouc [5] (see also [17]).

\[ \cdots \to \text{Ind} \tilde{H}_i C_{n-4}^d \to \tilde{H}_i C_{n-1}^d \to \text{Res} \tilde{H}_i C_n^d \to \text{Ind} \tilde{H}_{i-1} C_{n-4}^d \to \cdots \]

where \( \text{Ind} = \text{Ind}_{S_{n-1}^d \times S_3}^{S_n^d} \), \( \text{Res} = \text{Res}_{S_{n-4}^d}^{S_n^d} \).

Together with a result of Athanasiadis [2], which we will give a different proof in section 3, for \( n \leq 44 \) the matching complexes \( C_n^d \) has at most two non-zero homology groups. In particular, the Euler characteristic of the matching complex \( C_n^d \) is equal to

\[ \chi(C_n^d) = (-1)^i \tilde{H}_i + (-1)^{i+1} \tilde{H}_{i+1} \]

for \( i = \lfloor \frac{n-1}{2} \rfloor = 2 \). Moreover, it can be computed as the alternating sum of the spaces of \( i \)-faces of \( C_n^d \). By Shareshian and Wachs [23], the character of the space \( F_{i-1}(C_n^d) \) of \( i - 1 \)-faces of \( C_n^d \) as representation of \( S_n \) is given by:

\[ \text{ch} F_{i-1}(C_N^d) = e_i [h_d] h_{n-id}, \]
where $e$ (respectively $h$) denotes the elementary (respectively homogeneous) symmetric functions.

The computation of the character of the Euler characteristic of $C_4^4$ for $n \leq 43$ is made possible by Sage [24]. From the computation, we get the expected values $A$ and $B$ of the possibly non-zero homology groups of $C_n^4$. In other words, $\tilde{H}_iC_n^4 = A + C$ and $\tilde{H}_{i+1}C_n^4 = B + C$ for some representation $C$. To determine $C$, from the exact sequence (1.3), we deduce an exact sequence of the form

\begin{equation}
0 \to \text{Res} \to X \to X \to \text{Res} \to 0
\end{equation}

where $X$ is explicitly computed from the exact sequence and induction. From that, we deduce the possible irreducible subrepresentations of $C$. Now to determine which of these possible irreducible subrepresentations of $C$ do appear in $C$ we note the following. If $V^\lambda$ is a subpresentation of $C$, then $V^\lambda$ appears in both $\tilde{H}_iC_n^4$ and $\tilde{H}_{i+1}C_n^4$. In other words, the squarefree divisor simplicial complex corresponding to $\lambda$, defined below, has two non-trivial homology groups.

**Definition 5** (Squarefree divisor simplicial complex). For each $\lambda = (\lambda_1, \ldots, \lambda_l)$, let $\Delta(\lambda)$ be the simplicial complex on the vertices $\{v \in \mathbb{N}^l : |v| = 4\}$ such that $F = \{v_1, \ldots, v_j\}$ is a face of $\Delta(\lambda)$ if and only if

$$\lambda - \sum_{v_j \in F} v_j \geq 0$$

where $\lambda \geq 0$ means $\lambda_j \geq 0$ for all $j$.

The fact that when $\lambda$ appears in both $\tilde{H}_iC_n^4$ and $\tilde{H}_{i+1}C_n^4$ implies that $\tilde{H}_i(\Delta(\lambda))$ and $\tilde{H}_{i+1}(\Delta(\lambda))$ are non-trivial follows from a result of Bruns and Herzog [7]. This is also the underlying theme in the result of Karaguerian, Reiner and Wachs [16].

From the definition, we see that when $|\lambda|$ is not divisible by 4 then

$$\Delta(\lambda) = \bigcup_{i=1}^{l(\lambda)} \Delta_i,$$

where $l(\lambda)$ is the number of rows of $\lambda$, and $\Delta_i = \Delta(\lambda_i)$ where $\lambda_i$ is just obtained from $\lambda$ by subtracting 1 in the $i$th row. Note that $\lambda_i$ might not be a partition, but $\Delta(\lambda_i)$ is isomorphic to the $\Delta(\text{sort}(\lambda_i))$, where $\text{sort}(\lambda)$ corresponds to the partition with the same content as $\lambda$. Thus the homology groups of $\Delta(\lambda)$ can be determined from the Mayer-Vietoris sequence and induction. A simple computation procedure finishes the proof of the main theorem. We want to mention that this method gives an easier proof of the result of [25].

The paper is organized as follows. In section 2, we compute the Clifford index of complete intersection of two general quartics in $\mathbb{P}^3$. We then deduce the Ottaviani-Paoletti conjecture for the fourth Veronese embedding of $\mathbb{P}^3$. In section 3, we prove a theorem of Athanasiadis [2] saying that certain skeleta of matching complexes are shellable. Finally, in section 4, we prove the vanishings of homology groups of certain matching complexes, from that we deduce the main theorem.

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2. Clifford index and canonical curves

In this section, we briefly introduce the notion of Clifford index and Green’s conjecture and their connection to the Ottaviani-Paoletti conjecture. For more information, we refer to [8].

Let \( C \) be a smooth projective curve of genus \( g \geq 4 \). A line bundle \( L \) on \( X \) is called special if \( h^1(L) \neq 0 \). The Clifford index of \( L \) is defined to be
\[
\text{Cliff} \ L = \deg L - 2(h^0(L) - 1).
\]

**Definition 6.** The Clifford index of a smooth projective curve \( C \) of genus \( g \geq 4 \) is:
\[
\text{Cliff} \ C = \min \{ \text{Cliff} \ L | h^0(L) \geq 2 \text{ and } h^1(L) \geq 2 \}.
\]

A line bundle \( L \) is said to compute the Clifford index of \( C \) if \( \text{Cliff} \ L = \text{Cliff} \ C \).

Green’s conjecture says that the Clifford index of \( C \) determines the Green-Lazarsfeld index of the canonical line bundle of \( C \):

**Conjecture 7** (Green). Let \( C \) be a smooth nonhyperelliptic curve over a field of characteristic 0. Then
\[
p(C, \omega_C) = \text{Cliff} \ C - 1.
\]

Let us come back to the Ottaviani-Paoletti conjecture in the case of \( d \)th Veronese embedding of \( \mathbb{P}^3 \). Let \( C \) be the curve obtained by taking two general hyperplane sections. The Hilbert polynomial of \( C \) can be easily computed from the Hilbert polynomial of the embedding of \( \mathbb{P}^3 \). Precisely,
\[
H_C(k) = d^3k - (d^3 - 2d^2).
\]

It follows that \( \deg C = d^3 \) and \( g(C) = d^3 - 2d^2 + 1 \). In particular, when \( d > 4 \) the degree of the embedding is less than the degree of the canonical divisor of \( C \). The same phenomena happen when we consider embeddings of higher dimensional projective spaces. Thus most of recent geometric techniques do not work in understanding the syzygies of \( C \).

When \( d = 3 \), we have a curve of genus 10 and degree 27. Thus by the result of Green [12], the embedding satisfies property \( N_6 \) which is the result of Ottaviani and Paoletti [20]. When \( d = 4 \) which is the main focus of the paper, \( C \) is a canonical curve of genus 33. Moreover, \( C \) lies on a K3 surface. By a theorem of Aprodu and Farkas [1], \( p(\mathbb{P}^3, \mathcal{O}(4)) = \text{Cliff} \ C - 1 \). Thus to prove the Ottaviani-Paoletti conjecture in this case, it suffices to determine the Clifford index of \( C \). Since \( C \) lies on a K3 surface, its Clifford index can be computed by the following

**Theorem 2.1** (Green-Lazarsfeld [14]). Let \( L \) be a base point free line bundle on a K3 surface \( S \) with \( L^2 > 0 \). Then \( \text{Cliff} \ C \) is constant for all smooth irreducible \( C \in |L| \), and if \( \text{Cliff} \ C < \left\lfloor \frac{g-1}{2} \right\rfloor \), then there exists a line bundle \( M \) on \( S \) such that \( M_C = M \otimes \mathcal{O}_C \) computes the Clifford index of \( C \) for all smooth irreducible \( C \in |L| \).

We are now ready for a proof of the Ottaviani-Paoletti conjecture in the case of \( \mathbb{P}^3 \) and \( \mathcal{O}(4) \):

**Theorem 2.2.** The fourth Veronese embedding of \( \mathbb{P}^3 \) satisfies property \( N_9 \).

**Proof.** Let \( S \in |\mathcal{O}_{\mathbb{P}^3}(4)| \) be a general hyperplane section. By the adjunction formula, we see that \( S \) is a K3 surface. Let \( C \in |\mathcal{O}_{S}(1)| \) be a general hyperplane section. By Ottaviani and Paoletti [20], \( \text{Cliff} \ C \leq 10 \), therefore we are in the second situation of Theorem 2.1, since \( g(C) = 33 \). In other words, there exists a
line bundle $M$ on $S$ such that $M_C = M \otimes \mathcal{O}_C$ computes the Clifford index of $C$. Moreover, by Noether-Lefschetz Theorem [19] Pic $S \cong \mathbb{Z}\ell$ where $\ell$ is the class of the hyperplane section of $\mathbb{P}^3$. Therefore $\ell_C = \ell \otimes \mathcal{O}_C$ computes the Clifford index of $C$. Since $\ell_C$ corresponds to a $g^1_1$ on $C$, thus

$$\text{Cliff } C = \text{Cliff } \ell_C = 16 - 2 \cdot 3 = 10.$$  

The theorem follows from Aprodu-Farkas Theorem [1].

### 3. Shellability of skeleta of matching complexes

In this section, we prove the vanishings of certain homology groups of matching complexes by showing that certain skeleta of matching complexes are shellable. Let us first recall the notion of shellability of simplicial complexes.

**Definition 8.** Let $\Delta$ be a pure simplicial complex with the set of facets $\mathcal{F}$. A total ordering $>$ on $\mathcal{F}$ is said to be a shelling of $\Delta$ if for every facet $F$ which is not smallest with respect to $>$, we have $F \cap \mathcal{F}_{<F}$ is a pure simplicial complex of codimension 1 in $F$, where $\mathcal{F}_{<F}$ is the simplicial complex whose facets are the facets of $\Delta$ which is smaller than $F$ with respect to $>$. When $\Delta$ has a shelling, we say that $\Delta$ is shellable.

The following notation will be used throughout the section. For a finite ordered set $S$, we denote $\min S$ the smallest element in $S$. When $A = \{1, \ldots, n\}$, for any $t$, we order the $t$-element subsets of $A$ lexicographically. Suppose $B = \{\beta_1, \ldots, \beta_t\}$ is an ordered set. When we wish to indicate that the elements of $B$ are in the order $\beta_1 < \cdots < \beta_t$, then we write $B = \beta_1 \cdots \beta_t$ without commas and parenthesis. For example, if $F = \{a_1, \ldots, a_{k+1}\}$ is a $k$-face of $C^d_A$, then writing $F = a_1 \cdots a_{k+1}$ signifies that $a_1 < a_2 < \cdots < a_{k+1}$ in lexicographic ordering.

Let $F = a_1 \cdots a_{k+1}$ and $G = b_1 \cdots b_{k+1}$ be two $k$-faces of $C^d_A$. We say that $F$ is larger than $G$ in lexicographic ordering, denoted by $F > G$, if $a_i > b_i$ for the first index $i$ where $a_i \neq b_i$.

Finally, assume that $F = a_1a_2 \cdots a_{k+1}$ is a face of $C^d_A$. Let $x$ be the smallest element in the complement of $a_1 \cup \cdots \cup a_{k+1}$. We set $c(F) = \{a_1 \cup \{x\}\}$.

For example, set $n = 5$ and $d = 2$. Then $134 > 123$ as 3-element subsets of $A = \{1, 2, \ldots, 5\}$. In the matching complex $C^d_A$, the 1-face 13 24 is larger than the 1-face 12 45 in the lexicographic ordering. And finally, when $F = 13 \ 24$, we have $c(F) = 123$.

Note that a graph (pure 1-dimensional simplicial complex) is shellable if and only if it is connected. Therefore, the 1-skeleton of the matching complex $C^d_n$ is shellable when $n \geq 2d + 1$.

**Lemma 3.1.** Assume that $k \geq 2$ and that the $(k - 1)$-skeleton of $C^d_n$ is shellable. The $k$-skeleton of $C^d_{n+d+1}$ is shellable.

**Proof.** Let $A = \{1, \ldots, n + d + 1\}$. Fix a shelling $\succ_1$ of $(k - 1)$-skeleton of $C^d_A$ as in the assumption. For any $n$-element subset $B$ of $A$, we denote $\succ_B$ the shelling of $(k - 1)$-skeleton of $C^d_B$ coming from $\succ_1$ corresponding to the order-preserving bijection between $B$ and $\{1, \ldots, n\}$.

We define an ordering $\succ$ of the $k$-faces of $C^d_A$ as follows. Let $F = a_1 \cdots a_{k+1}$ and $G = b_1 \cdots b_{k+1}$ be two $k$-faces of $C^d_A$. Then $F \succ G$ if and only if

1. $1 \notin a_1$, and $F > G$ in the lexicographic ordering; or
(2) \(1 \in a_1, 1 \in b_1\), and \(c(F) > c(G)\); or
(3) \(1 \in a_1, 1 \in b_1, c(F) = c(G)\) and \(a_1 > b_1\); or
(4) \(1 \in a_1, 1 \in b_1, c(F) = c(G), a_1 = b_1,\) and \(a_2 \cdots a_{k+1} \supset B b_2 \cdots b_{k+1},\) where \(B\) is the complement of \(c(F)\).

We will now show that this is a shelling of the \(k\)-skeleton of \(C^d_n\). Let \(F = a_1 \cdots a_{k+1}\) be a \(k\)-face of \(C^d_n\). Furthermore, assume that \(F\) is not smallest with respect to the \(\succ\)-order. Let \(\mathcal{F}\) be the simplicial complex whose faces are all the \(k\)-faces of \(C^d_n\) that are less than \(F\) in the \(\succ\)-order. Let \(H = F \cap \mathcal{F}\) be a facet of \(F \cap \mathcal{F}\). We need to show that \(H\) has codimension 1 in \(F\).

Moreover assume that \(G = b_1 \cdots b_{k+1}\).

Case 1: \(1 \notin a_1\). Assume that \(a_i \notin H\) for some \(i\). Let \(b_1\) be any \(d\)-element subset containing 1 of the complement of \(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_{k+1}\). Let \(F' = b_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{k+1}\), then \(F \supset F'\), and \(F \cap F' = F \setminus \{a_i\}\). Since \(H\) is a facet of \(F \cap \mathcal{F}\), we have \(H = F \setminus \{a_i\}\) having codimension 1 in \(F\).

Case 2: \(1 \in a_1\) and \(c(F) > c(G)\). Assume that \(c(F) = a_1 \cup \{x\}\) and \(c(G) = b_1 \cup \{y\}\). There are two subcases:

Subcase 2a: \(a_1 = b_1\). Since \(c(F) > c(G)\), we have \(x > y\), which implies that \(y \in a_i\) for some \(i > 1\). Since \(y \in c(G)\), this implies that \(a_i \neq b_j\) for any \(j\). In other words, \(a_i \notin H\). Let \(a'_i = a_1 \setminus \{y\} \cup \{x\}\). Let \(F' = \{a'_1, \ldots, a_{i-1}, a'_i, a_{i+1} \cup \cdots \cup a_{k+1}\}\). Then \(c(F') = a_1 \cup \{y\} < c(F)\). Moreover \(F \cap F' = F \setminus \{a_i\}\) which is of codimension 1. Since \(a_i \notin H\), and \(H\) is a facet of \(F \cap \mathcal{F}\), \(F = F \cap F'\) is of codimension 1 in \(F\).

Subcase 2b: \(a_1 \neq b_1\). In this case, as \(1 \in a_1\) and \(1 \notin b_i\) for any \(i > 1\), we have \(1 \notin H\). If \(a_1\) is not the smallest \(d\)-element subset of \(c(F)\), let \(a'_1\) be the smallest such element. Let \(F' = a'_1 a_2 \cdots a_{k+1}\). We have \(F \supset F'\) and \(F \cap F' = a_2 \cdots a_{k+1}\) which is of codimension 1 in \(F\). Therefore, we may assume that \(a_1\) is the smallest \(d\)-element subset of \(c(F)\). This implies that \(x\) is larger than any element in \(a_1\). Let \(z\) be the smallest element in \(b_1 \setminus a_1\). Since \(c(F) > c(G)\), this implies that \(z < x\). Therefore, \(z\) must belong to \(a_i\) for some \(i \geq 2\), and \(a_i \notin H\). Again, let \(a'_i = a_i \setminus \{z\} \cup \{x\}\). Let \(F' = \{a_1, \ldots, a_{i-1}, a'_i, a_{i+1} \cup \cdots \cup a_{k+1}\}\), then we have \(c(F') = a_1 \cup \{z\} < c(F)\), and \(F \cap F' = F \setminus \{a_i\} > H\) which is a contradiction.

Case 3: \(1 \in a_1\), \(c(F) = c(G)\) and \(a_1 > b_1\). In this case \(a_1 \notin H\). Since \(c(F) = c(G)\), this implies that if we let \(F' = b_1 a_2 \cdots a_{k+1}\), then \(F \supset F'\) and \(F \cap F' = a_2 \cdots a_{k+1}\) which is of codimension 1. Therefore, \(H\) is of codimension 1 in \(F\).

Case 4: \(1 \in a_1\), \(c(F) = c(G)\) and \(a_1 = b_1\). In this case, by the shelling on the complement of \(c(F)\), \(H\) also has codimension 1 in \(F\).

As an application, we give another proof of a theorem of Athanasiadis [2].

**Theorem 3.2.** The \(k\)-skeleton of \(C^d_n\) is shellable when \(k \leq \frac{n+1}{2} - 1\).

**Proof.** For any \(n \geq 2d + 1\), the 1-skeleton of \(C^d_n\) is shellable. By Lemma 3.1, for any \(n \geq kd + k - 1\), the \(k - 1\)-skeleton of \(C^d_n\) is shellable. This finishes the proof of the theorem. \(\square\)

4. Homology of matching complexes

In this section, using the equivariant long exact sequence (1.3), Theorem 3.2 and induction, we compute homology groups of the matching complexes \(C^d_n\) for \(n \leq 44\). From that we deduce the main theorem. In the following, we denote the partition
\( \lambda \) with row lengths \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \) by the sequence \((\lambda_1, \lambda_2, \ldots, \lambda_k)\) and we use the same notation for the representation \( V^\lambda \). To simplify notation, we omit the subscript and superscript when we use the operators Ind and Res. It is clear from the context and the equivariant long exact sequence what the induction and restriction are.

Throughout this section \( d = 4 \). For simplicity, we sometimes write \( C_n \) for \( C_n^4 \).

We recall from the introduction the strategy to determine the homology groups of the matching complexes \( C_n^4 \). From the computation of the Euler characteristic of \( C_n^4 \) we get the expected values for the possibly non-trivial homology groups \( \tilde{H}_i(C_n^4) \) and \( \tilde{H}_{i+1}(C_n^4) \). By induction and the exact sequence (1.3), we deduce an exact sequence of the form (1.4). From that, we determine all possible values of subrepresentations of \( C_n^4 \). Now, let \( \lambda \) be a possible irreducible subrepresentation of \( C_n^4 \). To eliminate \( \lambda \), we show that either \( \tilde{H}_i(\Delta(\lambda)) = 0 \) or \( \tilde{H}_{i+1}(\Delta(\lambda)) = 0 \). To do so, we use induction and the Mayer-Vietoris sequence and note that when \( |\lambda| \) is not divisible by 4 then

\[ \Delta(\lambda) = \bigcup_{i=1}^{\lambda} \Delta_i, \]

where \( \Delta_i = \Delta(\lambda_i) \), and \( \lambda_i \) are obtained from \( \lambda \) by subtracting one from corresponding rows of \( \lambda \).

For simplicity of the statement, when \( C = 0 \), we say that the homology groups of \( C_n^4 \) are the expected ones. Sometimes we call \( C \) the supplementary representation of \( C_n^4 \). From the exact sequence (1.3) and induction, it is not hard to see the following.

**Proposition 4.1.** The homology groups of \( C_n^4 \) with \( n \leq 20 \) are the expected ones.

**Proposition 4.2.** The homology groups of \( C_{21}^4 \) are the expected one.

**Proof.** Applying the equivariant long exact sequence (1.3) with \( |A| = 21 \) we have an exact sequence

\[ \cdots \to \tilde{H}_3 C_{20} \to \text{Res} \tilde{H}_3 C_{21} \to \text{Ind} \tilde{H}_2 C_{17} \to 0. \]

From the computation of the Euler characteristic of \( C_{21}^4 \) we get the expected value for \( \tilde{H}_3 C_{21} \), called \( A \). Let \( \tilde{H}_3 C_{21} = A + C \). We have \( \tilde{H}_3 C_{20} + \text{Ind} \tilde{H}_2 C_{17} = \text{Res} A \) maps surjectively onto \( \text{Res} C \). Therefore, the only possible irreducible subrepresentations of \( C \) are

\[(7, 5, 5, 2, 2), (7, 6, 3, 3, 2) \text{ and } (8, 5, 3, 3, 2).\]

Let \( \lambda = (7, 5, 5, 2, 2) \). We will show that \( \lambda \) does not appear in \( C \) by showing that \( \tilde{H}_3(\Delta(\lambda)) = 0 \). Note that

\[ \Delta(\lambda) = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5 \]

where each \( \Delta_i \) has \( \tilde{H}_3(\Delta_i) = 0 \) by Proposition 4.1. Moreover, we have

\[ \Delta_{ij} = \Delta_i \cap \Delta_j \]

have trivial second homology groups by Proposition 4.1. By the Mayer-Vietoris sequence, \( \tilde{H}_3(\Delta(\lambda)) = 0 \). Thus \( \lambda \) does not appear in \( C \).

Similarly, \( (7, 6, 3, 3, 2) \) and \( (8, 5, 3, 3, 2) \) do not appear in \( C \). In particular \( C = 0 \), the proposition follows.

Processing in the same manner we deduce

**Proposition 4.3.** The homology groups of \( C_{22}^4 \) are the expected ones.
Proposition 4.4. The homology groups of $C_{23}^4$ are the expected ones.

Proof. By Theorem 3.2, it suffices to show that $\tilde{H}_3C_{23} = 0$. Applying the equivariant long exact sequence (1.3) with $|A| = 23$, we have $\tilde{H}_3C_{22}$ maps surjectively onto $\text{Res} \tilde{H}_3C_{23}$. By Proposition 4.3,

$$\tilde{H}_3C_{22} = (9, 9, 4) \oplus (10, 9, 3) \oplus (11, 9, 2) \oplus (12, 9, 1),$$

thus $\tilde{H}_3C_{23} = 0$. □

It is worth noting that the homology groups of the matching complexes $C_{4n}^4$ and $C_{4n}^3$ are easier to determine once we know the homology groups of the previous matching complexes. Note that by the result of Green, we know that the homology groups of $C_{24}^2$ are the expected ones already. Also, by the result of Bruns, Conca and Römer the homology groups of $C_{26}^4$ are the expected ones.

Proposition 4.5. The homology groups of $C_{25}^4$ are the expected ones.

Proof. The proof is similar to that of Proposition 4.2. □

For the matching complex $C_{26}^4$, it is quite surprising that its homology groups are not the expected ones. In other words, the naturality property fails for general matching complexes. This failure makes the determination of the homology groups of the matching complexes much more complicated.

Proposition 4.6. The only non-zero homology groups of $C_{26}^4$ are $\tilde{H}_4C_{26}^4$ and $\tilde{H}_5C_{26}^4$. Let $A$ and $B$ be the expected values of $\tilde{H}_4C_{26}$ and $\tilde{H}_5C_{26}$. We have $\tilde{H}_4C_{26} = A + C$ and $\tilde{H}_5C_{26} = B + C$ for some nontrivial representation $C$.

Proof. By Theorem 3.2, $\tilde{H}_iC_{26} = 0$ for $i \neq 4, 5$. Applying the equivariant long exact sequence (1.3) with $n = 26$, we see that $\text{Res} \tilde{H}_4C_{26}$ maps surjectively onto $\text{Ind} \tilde{H}_3C_{22}$. By Lemma 4.3,

$$\tilde{H}_3C_{22} = (9, 9, 4) \oplus (10, 9, 3) \oplus (11, 9, 2) \oplus (12, 9, 1).$$

From the computation of the Euler characteristic of $C_{26}^4$ we have

$$A = (9, 9, 7, 1) \oplus (10, 8, 8) \oplus (10, 9, 7) \oplus 2(10, 10, 6) \oplus (11, 8, 7) \oplus 2(11, 9, 5, 1)$$

$$\oplus (11, 9, 6) \oplus 2(11, 10, 5) \oplus (11, 11, 3, 1) \oplus 2(12, 8, 6) \oplus (12, 9, 4, 1)$$

$$\oplus 2(12, 9, 5) \oplus (12, 10, 2, 2) \oplus (12, 10, 3, 1) \oplus 3(12, 10, 4) \oplus (12, 11, 2, 1)$$

$$\oplus (12, 11, 3) \oplus (12, 12, 2) \oplus (13, 7, 6) \oplus (13, 8, 5) \oplus 2(13, 9, 3, 1) \oplus (13, 9, 4)$$

$$\oplus (13, 10, 2, 1) \oplus (13, 10, 3) \oplus (13, 11, 1, 1) \oplus (14, 6, 6) \oplus (14, 8, 4)$$

$$\oplus (14, 9, 2, 1) \oplus (14, 9, 3) \oplus (14, 10, 2) \oplus (15, 9, 1, 1).$$

Since $\text{Res} A$ cannot be mapped surjectively onto $\text{Ind} \tilde{H}_3C_{22}$, we see that $C$ is nontrivial. □

We want to remark that, the precise determination of homology groups of $C_{26}^4$ is difficult and not necessary for our purpose, as we can still be able to determine:

Proposition 4.7. The homology groups of $C_{27}^4$ are the expected ones.

Proof. By Theorem 3.2, it suffices to show that $\tilde{H}_4C_{27} = 0$. Applying the long exact sequence (1.3), we see that $\tilde{H}_4C_{26}$ maps surjectively onto $\tilde{H}_4C_{27}$. From the proof of proposition 4.6, we deduce a bound for $\tilde{H}_4C_{26}$. From that, we deduce the
possible irreducible subrepresentation of $\tilde{H}_4 C_{27}$. Now assume that $\lambda$ corresponds to a possible irreducible subrepresentation of $\tilde{H}_4 C_{27}$, it suffices to prove that $\tilde{H}_4 \Delta(\lambda) = 0$. We use argument similar to that in proposition 4.2. For example, let $\lambda = (8, 7, 5, 5, 2)$ be such a partition. We note that

$$\Delta(\lambda) = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5$$

where from proposition 4.6, $\tilde{H}_4 \Delta_i = 0$. Also, from proposition 4.5 we know that $\Delta_{ij} = \Delta_i \cap \Delta_j$ has $\tilde{H}_3 \Delta_{ij} = 0$. Since further intersections have no second homology groups, we deduce that $\tilde{H}_4 \Delta(\lambda) = 0$. The same argument works for other partitions that could possibly appear in $C$.

We want to remark that in the above proof, to conclude that $\tilde{H}_4 \Delta_i = 0$, we note that the partitions corresponding to $\Delta_i$ appear in the expected value of $\tilde{H}_5 C_{26}$ and thus do not appear in $\tilde{H}_4 C_{26}$. Since each time doing intersection, the size of the corresponding partition only drop down by 1, while we only need to prove the vanishing of the earlier homology groups, which is trivial. Thus the main observation is that $\tilde{H}_4 \Delta_i = 0$.

Before computing the homology groups of other matching complexes, we want to make a few comments about the computation. The code for computing the Euler characteristic of $C_4^n$ is simple. For example, to calculate Euler characteristic of $C_4^{25}$, in sage, we need to compute


where $s$ stands for the Schur symmetric functions, and $a_1, ..., a_6$ are the plethysms. For example, $a_6 = (e[6]).plethysm(s[4])$. To compute Euler characteristic of $C_4^n$ for $n \leq 43$ we need to compute the plethysms up to $a_{10} = e[10].plethysm(s[10])$. The only difficult computation is the plethysm, and the last one is the most complicated one. In Sage [24], if we write $s[1, 1]$ instead of $e[2]$, the computation will be given in Schur basis. Since we want the result to be in the Schur basis, we can input it that way. Nevertheless, in actual computation, at the last step the computation in Schur basis takes so long, we compute the plethysm in elementary symmetric function basis rather than Schur basis. We then divide the result in smaller part and changes to the Schur basis. From that we get the expected values for the homology groups of the matching complexes. Now, from the exact sequence and induction, we deduce the possible common subrepresentations of the two possibly non-zero homology groups of $C_4^n$. The expected values of homology groups of $C_4^n$ for $28 \leq n \leq 43$ as well as the possible irreducible subrepresentations of the supplementary representations of $C_{4k-1}$ for $8 \leq k \leq 11$ are posted at

http://www.math.unl.edu/ tvu5/expectedHomology.txt

In the same manner, though we do not completely determine the homology groups of $C_{29}^4, C_{30}^4$, we still see that $C_{31}^4$ has the expected homology groups. Precisely, $\tilde{H}_5 C_{31} = 0$. From that, we see that $\tilde{H}_5 C_{32} = 0$. In summary, we have

**Proposition 4.8.** The homology groups of $C_{31}^4, C_{32}^4, C_{35}^4, C_{36}^4, C_{39}^4, C_{40}^4$ are the expected ones.

We also have
Proposition 4.9. The only nonzero homology groups of $C_{43}^4$ are $\tilde{H}_8 C_{43}^4$ and $\tilde{H}_9 C_{43}^4$. Moreover, $\tilde{H}_9 C_{43}^4$ contains no irreducible subrepresentations corresponding to partitions with more than 4 rows.

Proof. From the long exact sequence (1.3) and Proposition 4.8, we deduce that $\tilde{H}_i C_{43}^4 = 0$ for $i < 8$. From the computation of the Euler characteristic of $C_{43}^4$, we deduce the expected values for the homology groups $\tilde{H}_8 C_{43}^4$ and $\tilde{H}_9 C_{43}^4$. Assume that $\tilde{H}_8 C_{43}^4 = A + C$ and $\tilde{H}_9 C_{43}^4 = B + C$. From the exact sequence of the form

$$0 \to \text{Res} C \to X \to X \to \text{Res} C \to 0$$

we deduce all possible irreducible subrepresentations of $C$. It suffices to show that if $\lambda$ has length at least 5, then $\lambda$ does not appear in $C$. For example, $\lambda = (10, 9, 9, 9, 6)$ is such a partition. We will show that $\tilde{H}_9 (\Delta(\lambda)) = 0$. Note that

$$\Delta(\lambda) = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5.$$

By Proposition 4.8, we deduce that $\tilde{H}_9 (\Delta_i) = 0$, and $\tilde{H}_8 (\Delta_{ij}) = 0$, where $\Delta_{ij} = \Delta_i \cap \Delta_j$. Similarly, we can check that either $\tilde{H}_9 (\Delta(\lambda)) = 0$ or $\tilde{H}_8 (\Delta(\lambda)) = 0$ for other partitions $\lambda$ with at least 5 rows that could possibly appear in $C$. □

We are now ready for the proof of the main theorem.

Theorem 4.10. The fourth Veronese embeddings of projective spaces satisfy property $N_9$.

Proof. Applying the long exact sequence (1.3), we have $\tilde{H}_i C_{44}^4 = 0$ for $i < 8$. Moreover, $\tilde{H}_8 C_{43}^4$ maps surjectively onto $\text{Res} \tilde{H}_8 C_{44}^4$. By Theorem 2.2, the fourth Veronese embedding of $P^3$ satisfies property $N_9$. In turn, by Karaguerian-Reiner-Wachs Theorem [16], $\tilde{H}_8 C_{44}^4$ does not contain any irreducible representations whose corresponding partitions have at most 4 rows. The theorem then follows from Proposition 4.9. □

References


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