LEVELS OF PERFECT COMPLEXES

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Abstract. A complex of \( R \)-modules \( F \) is perfect if it belongs to the smallest thick subcategory of \( D(R) \) that contains \( R \). The latter implies that every perfect complex can be built from \( R \) using a finite sequence of mapping cones, shifts, and direct summands. The level of \( F \), denoted by \( \text{level}^R(F) \), measures the number of mapping cones needed to build \( F \). In this paper, we prove that the level of a perfect complex is bounded below by the largest gap in its homology. This result allows us to give estimates for levels of Koszul complexes. As another consequence of our main result, we derive an improved version of the New Intersection Theorem for rings containing a field.

INTRODUCTION

Let \( R \) be an associative ring. A complex of \( R \)-modules is perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective \( R \)-modules. In this paper we focus on the level of perfect complexes. The level of a complex \( F \), denoted \( \text{level}^R(F) \), is the minimum number of mapping cones needed to build \( F \) from \( R \). Levels may are a generalization of the concept of projective dimension in the sense that \( \text{level}^R(M) = \text{proj dim}_R M + 1 \) for any \( R \)-module \( M \).

The notion of level was introduced and studied by Avramov, Buchweitz, Iyengar, and Miller in [2]. In their work, they use level to study invariants of complexes and rings such as Loewy lengths of homologies and the conormal free rank of \( R \) (see [2, Theorem 3]). What makes levels theoretically useful is the fact that it behaves well under operations commonly applied to perfect complexes. For instance level is well behaved under cone preserving functors, see Lemma [13].

Studying the number of steps it takes to “build” one object from another in a triangulated category is not new. Indeed, similar notions in algebraic topology were translated by Dwyer, Greenless, and Iyengar in [7] to the commutative setting in order to study perfect complexes. This notion was studied for general triangulated categories by Rouquier who, inspired by the work of Bondal and Van den Bergh in [6], defined in [13] the dimension of triangulated categories.

Algebraists have gained an intuitive understanding of many invariants in homological algebra such as depth, dimension, and projective dimension. However, level still appears to be a mysterious invariant and so far has eluded such understanding. The goal of this project is to discover what level actually measures. To that end we have investigated bounds on the level of a complex. Upper bounds for levels are relatively easy to obtain, since any explicit construction of a complex using...
mapping cones provides an upper bound. Therefore, this paper focuses on lower bounds.

The most general result of the paper is the following theorem, where we give a lower bound on the level of a perfect complex $F$ in terms of the largest gap in the homology of $F$.

**Theorem 2.1.** Let $F$ be a complex of $R$-modules. Assume $H_i(F) = 0$ for all $a < i < b$ with $a, b \in \mathbb{Z}$ and $H_0(\Omega_{b-1}^R(F))$ is not projective. Then

$$\text{level}^R(F) \geq b - a + 1.$$  

Recall that $\Omega_{b-1}^R(F)$ is the $(b-1)$th syzygy of $F$ (see Section 1).

The lower bound in Theorem 2.1 is sharp. Indeed, when $F$ is the projective resolution of a module $M$. If $\text{proj dim} M = n$, then $n$ is the largest value such that $H_0(\Omega_{n-1}^R(F)) = \Omega_{n-1}M$ is not projective. The theorem states that $\text{level}^R(F) \geq n$, however it is well known that in this case we have equality. Furthermore, the assumption that $H_0(\Omega_{b-1}^R(F))$ is not projective cannot be omitted, as can be seen by taking the complex

$$F := 0 \to R \to 0 \to 0 \to 0 \to R \to 0$$

which can have arbitrary large gap in homology but has level one.

As an application of Theorem 2.1, we establish lower bounds on the Koszul complex of an ideal $I$. In particular, we show that the level of the Koszul complex of the maximal ideal in a local ring is equal to the embedding dimension plus one. Our motivation for studying the Koszul complex is to understand levels in the context of other better understood invariants.

Restricting to commutative Noetherian local rings, as a consequence, we deduce an improved version of The New Intersection Theorem for rings containing a field.

**Theorem 4.1.** Let $R$ be a Noetherian local ring with a balanced big Cohen-Macaulay algebra $S$. Let

$$F := 0 \to F_n \to \cdots \to F_0 \to 0$$

be a complex of finitely generated free $R$-modules such that length $(H_i(F))$ is finite for every $i \geq 1$. For any ideal $I$ that annihilates a minimal generator of $H_0(F)$, there is an inequality

$$\text{level}^R(F) \geq \text{dim}(R) - \text{dim}(R/I) + 1.$$  

The result recovers a special case of theorem [2, 5.1] for commutative Noetherian rings. Indeed, we show in Theorem 4.2 that if $R$ contains a field and $I$ is the annihilator of $H(F)$, then

$$\text{level}^R(F) \geq \text{superheight}\ I + 1.$$  

The crux of the proof of Theorem 2.1 and other results in the paper, is to use the vanishing of homology to construct a sequence of Ghost maps and then applying the Ghost Lemma, Lemma [1.2].

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1. Preliminaries

Let $R$ be an associative ring. We denote by $\mathcal{D}(R)$ the derived category of
the category of $R$-modules. In this paper, a complex of $R$-modules will have the form

$$F = \cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \longrightarrow \cdots .$$

Given a complex $F$, the $k$th suspension of $F$ is the complex $\Sigma^k F$ with

$$(\Sigma^k F)_n = F_{n-k} \quad \text{and} \quad \partial^{\Sigma^k F} = (-1)^k \partial F.$$

For every $i \in \mathbb{Z}$, let $F_{\geq i}$ and $F_{\leq i}$ denote the hard truncations

$$F_{\geq i} := \cdots \longrightarrow F_{i+2} \xrightarrow{\partial_{i+2}} F_{i+1} \xrightarrow{\partial_{i+1}} F_i \longrightarrow 0 ,$$

$$F_{\leq i} := 0 \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} F_{i-2} \longrightarrow \cdots ,$$

with $F_i$ in homological degree $i$. Note that there is a natural projection $\tau^i : F \to F_{\geq i}$
where $\tau^i_n$ the identity for $n \geq i$ and zero otherwise. We will identify an $R$-module
$M$ with the complex concentrated in degree 0. We write $X \simeq Y$ if complexes $X$ and
$Y$ can be related through a sequence of quasi-isomorphisms, i.e., morphisms which
induce an isomorphism in homology. A complex $F$ is perfect if it is quasi-isomorphic
to a bounded complex of finitely generated projective modules. Equivalently, $F$ is
perfect if it is contained in $\text{thick}(R)$, the thick closure of $R$, i.e., the smallest thick
subcategory of $\mathcal{D}(R)$ containing $R$. For more information on triangulated categories
and thick subcategories, see [11].

We will now describe a filtration of $\text{thick}(R)$ that will allow us to associate a
numerical invariant to every perfect complex. For a more general version of this
construction for arbitrary thick closures, see [2, Section 2]. Given a subcategory $A$
of $\mathcal{D}(R)$, we say $A$ is strict if it is closed under isomorphisms in $\mathcal{D}(R)$. We write
$\text{add}(A)$ for the intersection of all strict and full subcategories of $\mathcal{D}(R)$ containing $A$
that are closed under finite direct sums and suspensions. We write $\text{smd}(A)$ for the
intersection of all strict and full subcategories of $\mathcal{D}(R)$ containing $A$ that are closed
under direct summands. For two strict and full subcategories $A$ and $B$ of $\mathcal{D}(R)$, we
denote by $A \ast B$ the full subcategory whose objects are

$$A \ast B = \left\{ M \mid \begin{array}{c}
L \longrightarrow M \longrightarrow N \longrightarrow \Sigma L \text{ is an exact triangle} \\
\text{with } L \in A \text{ and } N \in B
\end{array} \right\} .$$

We define

$$\text{thick}^0(R) = \{ 0 \}, \quad \text{thick}^1(R) = \text{smd}(\text{add}(R)), \quad \text{and}$$

$$\text{thick}^n(R) = \text{smd}(\text{thick}^{n-1}(R) \ast \text{thick}^1(R)), \quad \text{for every } n \geq 2 .$$

The full subcategory $\text{thick}^n(R)$ is called the $n$th thickening of $R$. It is clear that
$\text{thick}^n(R) \subseteq \text{thick}^{n+1}(R)$ for every $n \geq 0$, and it can be shown that $\bigcup_{n \geq 0} \text{thick}^n(R) = \text{thick}(R)$; see [7, 3.7]. Therefore, we obtain the following filtration:

$$\text{thick}^0(R) \subseteq \text{thick}^1(R) \subseteq \text{thick}^2(R) \subseteq \cdots \subseteq \text{thick}(R) .$$
Following [2], to each \( F \in \mathcal{D}(R) \) associate the number
\[
\text{level}^R(F) := \inf\{n \in \mathbb{N} \mid F \in \text{thick}^n(R)\},
\]
to the level of \( F \) over \( R \). We set \( \text{level}^R(F) = \infty \) if \( F \) is not in \( \text{thick}(R) \). We can think of the elements in \( \text{thick}(R) \) as being built from \( R \), and \( \text{level}^R(F) \) as the number of steps it takes to build \( F \). We have the following upper bound for levels:

**Lemma 1.1 ([2] 2.5.2).** Given a perfect complex
\[
F := 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_m \longrightarrow 0,
\]
we have \( \text{level}^R(F) \leq n - m + 1 \).

However, this is a poor bound in general: for instance, over a regular local ring \( R \), we have \( \text{level}^R(F) \leq \dim R + 1 \) for every perfect complex \( F \) (see [2] 5.4).

A chain map of complexes \( f: X \longrightarrow Y \) is said to be ghost if the map induced by \( f \) in homology, \( H(f): H(X) \longrightarrow H(Y) \), is the zero map. Note that \( \text{Hom}_{\mathcal{D}(R)}(R,F) = \text{Hom}_R(f,0), \) so a map \( f \) is ghost if \( \text{Hom}_{\mathcal{D}(R)}(\Sigma^k R, f) = 0 \) for all \( k \). The following result, known as the Ghost lemma, will be used in the proofs of the main results.

**Lemma 1.2 (Ghost lemma [3] 2.2).** Let
\[
X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n
\]
be a sequence of ghost maps. If a complex of \( R \)-modules \( F \) is such that the induced map
\[
\text{Hom}_{\mathcal{D}(R)}(F,f_{n-1} \circ \cdots \circ f_0): \text{Hom}_{\mathcal{D}(R)}(F,X_0) \longrightarrow \text{Hom}_{\mathcal{D}(R)}(F,X_n)
\]
is non-zero, then \( \text{level}^R(F) \geq n + 1 \).

The following is a basic fact about levels.

**Lemma 1.3.** Let \( R \) and \( S \) be associative rings, and \( \varphi: \mathcal{D}(R) \longrightarrow \mathcal{D}(S) \) an exact functor with \( \varphi(R) = S \). Then for every \( F \) in \( \mathcal{D}(R) \), we have
\[
\text{level}^S(\varphi(F)) \leq \text{level}^R(F).
\]

**Proof.** Since \( \varphi(A \star B) \subseteq \varphi(A) \star \varphi(B) \) for any \( A, B \in \mathcal{D}(R) \), it is easy to see that \( \varphi(\text{thick}^n(R)) \subseteq \text{thick}^n(S) \) for all \( n \in \mathbb{N} \). \( \square \)

Assuming \( H_{i<0}(F) = 0 \) and choosing a quasi-isomorphism \( F \simeq P \) with \( P \) a complex of projective modules that is bounded below, we denote the \( n \)th syzygy module of \( F \) for every \( n \in \mathbb{Z} \) by \( \Omega^n(F) := \Sigma^{-n}(P_{\geq n}) \). We can analyze how \( \Omega^n(F) \) depends on the choice of \( P \) via the following proposition, which \( \text{is [3] Lemma 1.2.} \)

**Proposition 1.4.** Let \( P \) and \( Q \) be quasi-isomorphic bounded below complexes of projective \( R \)-modules. Then for every \( n \in \mathbb{Z} \) there exist projective \( R \)-modules \( P' \) and \( Q' \) such that
\[
P_{\geq n} + \Sigma^n Q' \quad \text{and} \quad Q_{\geq n} + \Sigma^n P'
\]
are homotopy equivalent. In particular, for any bounded below complex \( F \) of \( R \)-modules, \( H_0(\Omega^n(F)) \) is unique up to projective summands.

Hence the condition that \( H_0(\Omega^n(F)) \) is projective is well defined since it is independent of the projective resolution used to compute \( \Omega^n(F) \).
2. Gaps and Levels

In this section, \( R \) will be an associative ring. The following is the main theorem in this section.

**Theorem 2.1.** Let \( F \) be a complex of \( R \)-modules. Assume \( H_i(F) = 0 \) for all \( a < i < b \) with \( a, b \in \mathbb{Z} \) and \( H_0(\Omega^R_{b-1}(F)) \) is not projective. Then

\[
\text{level}^R(F) \geq b - a + 1.
\]

**Proof.** If \( F \notin \text{thick}(R) \), the result is clear. Hence, we may replace \( F \) with a bounded complex of projective \( R \)-modules. Set \( Z_i(F) := \ker(\partial_i^F) \) for each \( i \), and consider the complex

\[
G := 0 \rightarrow Z_{b-1}(F) \rightarrow F_{b-1} \rightarrow F_{b-2} \rightarrow \cdots.
\]

By construction \( H_i(G) = H_i(F) \) for \( a < i < b \), and since \( Z_{b-1}(F) \rightarrow F_{b-1} \) is injective, we have \( H_b(G) = 0 \). Therefore, \( H_i(G) = 0 \) for all \( i > a \). Hence, each of the natural projections in the following sequence

\[
G \rightarrow G_{\geq a+1} \rightarrow G_{\geq a+2} \rightarrow \cdots \rightarrow G_{\geq b}
\]

is ghost.

We claim that the natural composition \( \varphi : F \rightarrow G \rightarrow G_{\geq b} \) given by

\[
\cdots \rightarrow F_{b+1} \rightarrow F_b \rightarrow F_{b-1} \rightarrow \cdots
\]

is non-zero in \( D(R) \) (i.e., it is not null-homotopic) and hence the result will follow by the *Ghost lemma* (Lemma 1.2). Equivalently, we claim there is no map \( \alpha : F_{b-1} \rightarrow Z_{b-1}(F) \) making the following diagram commute:

\[
\begin{array}{ccc}
F_b & \rightarrow & F_{b-1} \\
\downarrow \varphi_b & & \downarrow \\
Z_{b-1}(F) & \rightarrow & 0
\end{array}
\]

Since \( F_b \rightarrow F_{b-1} \) factors through \( F_b/Z_b(F) \cong Z_{b-1}(F) \), the existence of such a map \( \alpha \) would produce the splitting

\[
0 \rightarrow Z_{b-1}(F) \rightarrow F_{b-1} \rightarrow F_{b-1}/Z_{b-1}(F) \rightarrow 0,
\]

implying \( F_{b-1}/Z_{b-1} \) is projective. However, \( F_{b-1}/Z_{b-1}(F) = H_0(\Omega^R_{b-1}(F)) \), thus the existence of \( \alpha \) contradicts our assumption that \( H_0(\Omega^R_{b-1}(F)) \) is not projective, proving the claim.

\( \square \)

We can recover the following well known result.

**Corollary 2.2.** If \( M \) is a finitely generated \( R \)-module of finite projective dimension, then

\[
\text{level}^R(M) = \text{proj dim}(M) + 1.
\]
Corollary 2.3. Assume

\[ \text{The result follows.} \]

Fix

\[ \text{Proof.} \]

\[ \eta \text{ and let } F \exists \text{ a complex } \]

\[ \text{be a projective resolution of } M. \]  

By Lemma [1.1] we only need to show that level\(R\)(\(F\)) \(\geq n + 1\). Since \(\text{proj dim}(M) = n\), then \(H_0(\Omega^R_{n-1}(F))\) is not projective. Clearly, \(H_i(F) = 0\) for \(0 < i < n\), hence, by Theorem [2.1] level\(R\)(\(F\)) \(\geq n + 1\). The result follows. \(\square\)

3. Koszul complexes over local rings

Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring. Given an ideal \(I\) in a ring \(R\), we denote the Koszul complex on a minimal set of generators of \(I\) by \(K(I)\). The goal of this section is to find estimates for the level of \(K(I)\).

Recall that a complex \(F\) is said to be minimal if \(\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}\) for every \(i\).

Proposition 3.1. Let \(F\) be a complex of finitely generated \(R\)-modules. Assume there exists a map of complexes \(\eta : F \rightarrow G\) where \(G\) is a minimal complex such that \(H_i(G) = 0\) for all \(a < i < b\), and \(\text{Im}(\eta_b) \not\subset \mathfrak{m}G_b + Z_b(G)\). Then

\[ \text{level}_R(F) \geq b - a + 1. \]

Proof. If \(F \not\in \text{thick}(R)\) the result is clear. We may assume \(F\) is a minimal complex; see [12, 2.4]. The rest of the proof is similar to that of Theorem [2.1]. Consider the complex

\[ G' := 0 \rightarrow G_b/\mathfrak{m}G_b \rightarrow G_{b-1} \rightarrow \cdots \]

and let \(\eta' : F \rightarrow G'\) be the composition of \(\eta\) and the natural chain map from \(G\) to \(G'\). Notice \(H_i(G') = 0\) for all \(a < i\). Consider \(\varphi : F \rightarrow G'_{\geq b}\) given by the composition of \(\eta'\) and the following \(b - a\) ghost maps

\[ G' \rightarrow G'_{\geq a+1} \rightarrow G'_{\geq a+2} \rightarrow \cdots \rightarrow G'_{\geq b} \]

Since \(F\) and \(G\) are minimal, the existence of a chain homotopy from \(\varphi\) to the zero map implies \(\text{Im}(\varphi_b) = \text{Im}(\eta'_b) \subset \mathfrak{m}G'_b\). But this implies that \(\text{Im}(\eta_b)\) is contained in \(\mathfrak{m}G_b + Z_b(G)\), contradicting the assumptions. Therefore \(\varphi\) is non-zero in \(D(R)\) and the result follows by Lemma [1.2]. \(\square\)

Remark 3.2. Note that if we assume that \(H_i(G) = 0\) for all \(a < i \leq b\), then \(\text{Im}(\eta_b) \not\subset \mathfrak{m}G_b\) is equivalent to \(\text{Im}(\eta_b) \not\subset \mathfrak{m}G_b + Z_b(G)\).

Recall that \(\text{Tor}^R(R/I, k)\) is a graded commutative \(k\)-algebra (see [11, 2.3.2]). Therefore, there is a natural map of \(k\)-algebras

\[ \kappa^I : \bigwedge \text{Tor}^R_1(R/I, k) \rightarrow \text{Tor}^R(R/I, k). \]
Corollary 3.3. For an ideal $I$, we have an equality
\[
\text{level}^R(K(I)) \geq \sup \{ b \mid \kappa_b^I \neq 0 \} + 1.
\]

Proof. Let $G$ be a minimal free resolution of $R/I$. The map $K(I) \rightarrow R/I$ extends to a map of complexes $\eta : K(I) \rightarrow G$. Notice $K(I) \otimes_R k \cong \bigwedge \text{Tor}^R_1(R/I, k)$, $G \otimes_R k \cong \text{Tor}^R(R/I, k)$, and $\kappa = \eta \otimes k$. Therefore, $\kappa_b^I \neq 0$ is equivalent to $\text{Im}((\eta)_b) \not\subseteq \mathfrak{m}G_b$. The conclusion now follows from Proposition 3.1 and Remark 3.2.

Corollary 3.4.

(1) $\text{level}^R(K(m)) = \text{edim}(R) + 1$.

(2) If $R$ contains a field. If $I$ is generated by a system of parameters, then $\text{level}^R(K(I)) = \dim(R) + 1$.

Proof. Let $G$ be a minimal resolution of $k$. By Lemma 1.1 for part (1) we only need to prove $\text{level}^R(K(m)) \geq \text{edim}(R) + 1$. Recall from [1, 6.3.5] that $G$ is the acyclic closure of $k$, that is $G$ is a DG-algebra obtained from $R$ by adjoining exterior and divided power variables in a minimal fashion. In this construction, the DG-subalgebra generated by homogenous elements of $G$ in homological degree one is isomorphic to $K(m)$. Therefore, we have an injection $\eta : K(m) \hookrightarrow G$ and this injection splits as an $R$-module homomorphism. Therefore $\eta \otimes k$ is also an injection, and in particular $\eta_b \otimes k$ is non-zero. Furthermore, since $\text{Tor}^R_1(k, k) \cong G_1 \otimes k$, it follows from the previous discussion that $K(m) \cong \bigwedge \text{Tor}^R(k, k)$ and under this isomorphism, $\kappa$ can be identified with $\eta \otimes k$. Therefore, $\kappa_b \neq 0$, and so $\text{level}^R(K(m)) \geq \text{edim}(R) + 1$ follows from Corollary 3.2.

Part (2) follows from Lemma 1.1, Corollary 3.3, and the Canonical Element Conjecture (see [3, 2.7]).

The following proposition shows that in general, despite what Corollary 3.4 might suggest, $K(I)$ is not equal to $\mu(I) + 1$, where $\mu(I) = \dim_k(I \otimes_R k)$.

Proposition 3.5. If $R$ contains a field, then for any ideal $I \subseteq R$ and any $c \in \mathbb{N}$ there is an inequality
\[
\text{level}^R(K(I^c)) \leq \mu(I) + 1.
\]

Proof. Let $r_1, \ldots, r_n$ be a minimal generating set of $I$. Let $S = k[x_1, \ldots, x_n]$, $n = (x_1, \ldots, x_n)$ and consider the map $f : S \rightarrow R$ defined by $f(x_i) = r_i$. Note that $K(I^c, R) = K(n^c, S) \otimes_S R = K(n^c, S) \otimes_S R$. Hence, by Lemma 1.3 we have
\[
\text{level}^R(K(I^c, R)) \leq \text{level}^R(K(n^c, S)) = n + 1,
\]
where the last equality holds by [2, 5.3]. The result follows.

A complex $F$ is said to be indecomposable in $D(R)$, if given any quasi-isomorphism $F \simeq A \oplus B$ with $A$ and $B$ complexes, then either $A \simeq 0$ or $B \simeq 0$.

Proposition 3.6. Let $F$ be a minimal indecomposable perfect complex. If $F_b \neq 0$, then $\text{H}_b(\Omega^R_{F \rightarrow F})$ is not projective. In particular, if there exists an $a$ such that $H_a(F) = 0$ for all $a < i < b$, then $\text{level}^R(F) \geq b - a + 1$. 

Proof. By Theorem 2.1, we need only to prove the first statement. Suppose $H_0(\Omega^R_{b-1}(F))$ is projective. Let $\partial$ denote the differential of $F$. The complex $\Omega^R_{b-1}F$ is
\[ \cdots \to F_{b+1} \to F_b \xrightarrow{\partial_b} F_{b-1} \to 0 \to \cdots. \]
Therefore, the sequence
\[ F_b \xrightarrow{\partial_b} F_{b-1} \to H_0(\Omega^R_{b-1}(F)) \to 0 \]
is exact. Since $H_0(\Omega^R_{b-1}(F))$ is projective, $\partial_b$ splits. But then we have $F_{b-1} = \text{Im} \partial_b \oplus H_0(\Omega^R_{b-1}(F))$, and $F \cong A \oplus B$ where $A$ and $B$ are the complexes
\[ A = \cdots \to F_{b+1} \to F_b \xrightarrow{\partial_b} \text{Im} \partial_b \to 0 \to \cdots \]
\[ B = \cdots \to 0 \to H_0(\Omega^R_{b-1}(F)) \to F_{b-2} \to F_{b-3} \cdots \]
Since $F$ is indecomposable, either $A \simeq 0$ or $B \simeq 0$. However, since $F$ is minimal, then so is $A$ and $B$. Thus since any quasi-isomorphism of bounded minimal complexes of free modules is an isomorphism (see [1, 1.1.2]), either $A$ or $B$ is 0. Since $F_0 \neq 0$, $A$ cannot be zero. Therefore, $B$ must be zero, which implies that $H_0(\Omega^R_{b-1}(F))$ is zero too. Therefore, $\text{Im} \partial_b = F_{b-1}$, which contradicts the minimality of $F$. 

Together with Proposition 3.5, the following result shows that there are arbitrarily large indecomposable perfect complexes with small level.

**Proposition 3.7.** Suppose $(R, m, k)$ is local. For any ideal $I$, the complex $K(I)$ is indecomposable in $D(R)$.

**Proof.** Pick a minimal generating set $x := x_1, \ldots, x_n$ of $I$, and set $K := K(I)$. Suppose that $K \simeq A \oplus B$ where $A$ and $B$ are perfect complexes with non-zero homology. By [12, 2.4], we may assume $A$ and $B$ are both minimal, and hence so is $A \oplus B$. Any quasi-isomorphism of bounded minimal complexes of free modules is an isomorphism (see [1, 1.1.2]). Therefore, we may assume that $K = A \oplus B$ with $A$ and $B$ non-zero. In particular, $R = K_0 = A_0 \oplus B_0$. Since $R$ is indecomposable as an $R$-module, we must have $A_0 = 0$ or $B_0 = 0$.

Suppose without loss of generality that $B_0 = 0$ and consider $a = \min \{|i| B_i \neq 0\}$. Since $\text{Im} \partial^B_i = 0$, there exists $v \in K_a$ such that $v \notin mK_a$ and $\partial^B_a(v) = 0$. On the other hand, fixing bases for $K_a$ and $K_{a-1}$, the non-zero entries in the matrix corresponding to $\partial^K_a$ are equal to $\pm x_i$ for some $i$. Hence, $\partial^K_a(v) = 0$ corresponds to a nontrivial relation between the elements $x_1, \ldots, x_n$ modulo $m$. The latter contradicts the minimality of the set $\{x_1, \ldots, x_n\}$. We conclude $K$ is indecomposable, as desired. 

**Corollary 3.8.** For any ideal $I \subseteq R$, there is an inequality
\[ \text{level}^R(K(I)) \geq \text{depth}(I, R) + 1. \]
In particular, if $R$ contains a field and $I$ is a generated by a regular sequence, then for all $c \in \mathbb{N}$, we have
\[ \text{level}^R(K(I^c)) = \mu(I) + 1. \]

**Proof.** Set $n = \mu(I)$, hence $K(I^c)$ is not projective, and by Proposition 3.7, $K(I)$ is indecomposable. Since $H_i(K(I)) = 0$ for all $n - \text{depth}(I, R) < i \leq n$, Proposition 3.6 implies that
\[ \text{level}^R(K(I)) \geq n - (n - \text{depth}(I, R)) + 1 = \text{depth}(I, R) + 1. \]
The second statement follows from the first statement and Proposition 3.5. Indeed, these facts yield
\[ n + 1 \geq \text{level}^R(K(I^c)) \geq \text{grade}(I^c) + 1 = \text{depth}(I, R) + 1 = n + 1. \]

\[ \square \]

4. The Improved New Intersection Theorem for levels

Recall that an \( R \)-algebra \( S \) is said to be a balanced big Cohen-Macaulay algebra if every system of parameters of \( R \) forms an \( S \)-regular sequence. See [5, Chapter 8] for further reading.

Theorem 4.1. Let \( R \) be a Noetherian local ring with a balanced big Cohen-Macaulay algebra \( S \). Let
\[
F := 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0
\]
be a complex of finitely generated free \( R \)-modules such that \( \text{length}(H_i(F)) \) is finite for every \( i \geq 1 \). For any ideal \( I \) that annihilates a minimal generator of \( H_0(F) \), there is an inequality
\[
\text{level}^R(F) \geq \dim(R) - \dim(R/I) + 1.
\]

Proof. To begin, it suffices to consider the case when \( \dim(R) - \dim(R/I) \geq 1 \). Also, replacing \( F \) by its minimal free resolution, we can assume that \( F \) is minimal. Set \( G := F \otimes_R S \) and set
\[ s := \sup \{ i \mid H_i(G) \neq 0 \}. \]

Now, the proof of [10, 3.1] yields an inequality
\[
(4.1.1) \quad s \leq n - \dim R + \dim(R/I).
\]
In particular, \( s \leq n - 1 \).

Claim. The \( S \)-module \( \Omega := H_0(\Omega^S_{n-1}(G)) \) is not projective.

Indeed, since \( s \leq n - 1 \) the complex \( 0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow 0 \), with \( G_{n-1} \) in degree 0, is a free resolution of \( \Omega \). Since \( S \) is a big Cohen-Macaulay algebra we have \( mS \neq S \). Hence, there exists a maximal ideal \( n \) of \( S \) containing \( mS \). By the minimality of \( F \), one gets that \( \text{Im}(\partial G_n^c) \subseteq mG_{n-1} \). Therefore,
\[
\text{Tor}_1^S(S/n, \Omega) \cong (S/n) \otimes_S G_n \neq 0.
\]
This implies that \( \Omega \) is not flat, which justifies the claim.

Given the preceding claim, Theorem 2.1 yields the second inequality below
\[
\text{level}^R(F) \geq \text{level}^S(G) \geq n - s + 1.
\]
The first inequality is Lemma 1.3. It remains to recall (4.1.1). \( \square \)

This result can be used to recover a special case of the The New Intersection Theorem for DG-Algebras in [2]. Recall
\[
\text{superheight} I = \sup \{ \text{height} IT \mid T \text{ is a Noetherian } R\text{-algebra} \}.
\]

Theorem 4.2 ([2, Theorem 5.1]). Assume \( R \) contains a field. Let \( F \) be a perfect complex and let \( I \subset R \) the annihilator of \( \bigoplus_{i \in \mathbb{Z}} H_i(F) \). Then we have
\[
\text{level}^R(F) \geq \text{superheight} I + 1.
\]
Proof. Let $R \to T$ be a Noetherian $R$-algebra; in particular $T$ also contains a field. Let $p \in \text{Spec} T$ be minimal over $IT$. Therefore, by [9, Theorem 1], $T_p$ has a balanced Cohen-Macaulay algebra $S$. Furthermore, $IT_p$ is the annihilator of $\bigoplus_{i \in \mathbb{Z}} H_i(F \otimes T_p)$, hence $H_i(F \otimes T_p)$ has finite length for every $i$. From Lemma 1.3 and Theorem 4.1 we obtain

$$\text{level}^R(F) \geq \text{level}^{T_p}(T_p \otimes_R F) \geq \dim T_p - \dim T_p/IT_p + 1 = \text{height} p + 1.$$ 

The result follows. □

We have the inequality

$$\text{superheight} I \geq \text{height} I \geq \text{depth}(I, R).$$

In various situations, these invariants provide lower bounds for $\text{level}^R(K(I))$. Indeed, in the general case, Corollary 3.8 tells us that $\text{level}^R(K(I)) \geq \text{depth}(I, R) + 1$. When $R$ contains a field, Theorem 4.2 gives the largest of these bounds: $\text{level}^R(K(I)) \geq \text{superheight} I + 1$.

References

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