Flows, dynamics and algorithms for 3–manifold groups

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Groups and the word problem

Let $G = \langle A \mid R \rangle$ be a finitely presented group. The **word problem (WP)** for $G$ asks if there is an algorithm that determines whether a word $w \in A^*$ represents the identity in $G$.

**Theorem (Boone & Novikov)**

*Not every fin. pres. group has a decidable WP.*

**Classes of f.p. groups with solvable WP**

- Abelian groups
- Hyperbolic groups (via Dehn’s algorithm)
- Nilpotent and polycyclic groups (closure under extension)
- Fundamental groups of 3-manifolds (via geometrization)

**Goal:** *Common* algorithm for 3-manifold groups, solving the WP using finite state automata (bounding computational complexity)
Trees and flow functions

Take $G = \langle A \mid R \rangle$ a finitely presented group, and $\Gamma = \Gamma(G, A)$ its Cayley graph.

Ideas on effectively solving the WP

1. Find a spanning tree $T$ for $\Gamma$
2. Find a way to rewrite edges outside of $T$ to paths “closer to $T$”
3. Make sure that everything is computable (by FSA)

Example.

$BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$

$T :$ Down-horizontal-up
More Formally

Definition
A function $\Phi$ which takes edges in $\Gamma$ to paths in $\Gamma$ is called a *bounded flow function* for $(G, T)$ if there exists a constant $K > 0$ so that:

1. for $e \in T$, $\Phi(e) = e$
2. for every edge $e$, $\Phi(e)$ has length at most $K$
3. $\Phi(e)$ starts and ends at the same vertices as $e$
4. there is no infinite sequence $\{e_i\}$ so that $e_i \neq e_{i+1}$ and $e_{i+1}$ is an edge on $\Phi(e_i)$

A group with a bounded flow function is called *stackable*. If $\Phi$ can be computed by an FSA, then $G$ is called *autostackable*.

Notes
▶ Autostackable groups have WP solvable using only an FSA
▶ Autostackability is equivalent to the existence of a regular, bounded, prefix-sensitive rewriting system
More Formally

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**Notes**

- Autostackable groups have WP solvable using only an FSA
- Autostackability is equivalent to the existence of a regular, bounded, prefix-sensitive rewriting system
Example: $\mathbb{Z}^2$
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Other WP solutions by FSA’s

**Thm.** (Brittenham–Hermiller–Holt, 2014)

(Asynchronously) Automatic groups on prefix-closed (unique) normal forms are autostackable.

**Thm.** (BHH, 2014)

If $G$ has a finite convergent rewriting system, then $G$ is autostackable.
A consequence of the Geometrization Theorem (Perelman) for 3-manifolds:

**Cor.**

If $M$ is a compact 3-manifold with incompressible toral boundary, then $\pi_1(M)$ has solvable word problem.
3-manifold groups: Earlier word problem solutions

Types of Solutions

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<td>Hermiller–Shapiro 1999:</td>
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<td>ECHLPT 1992; Thurston 1992; N. Brady 2001: $M$ with no Nil or Sol pieces is automatic, otherwise not</td>
<td>$M$ geometric and not hyperbolic has a finite complete rewriting system</td>
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<td>Bridson–Gilman 1996: All closed $M$ have an asynchronous combing by an indexed language</td>
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**Cor.** (Brittenham–Hermiller–Holt 2014)

*If $M$ is a closed geometric 3-manifold, then $\pi_1(M)$ is autostackable.*
Main Theorem

**Thm.** (Brittenham–Hermiller–S. 2016)
If $M$ is a compact 3-manifold with incompressible toral boundary, then $\pi_1(M)$ is autostackable.

**Rmk.** This gives a common solution to the word problem using FSA’s.
Strategy

1. Break down $M$ and $\pi_1(M)$ into pieces that are “manageable.”
2. Prove that each of the pieces is autostackable.
3. Prove a combination theorem to glue the autostackable structures back together.

Note.
Every time we do an operation to simplify $M$, we need a theorem to tell us that autostackability of $\pi_1(simplified manifold)$ implies autostackability of $\pi_1(M)$. 
Decomposing $M$: Part I

**Orientation:**

**Topology:** Replace $M$ by its orientable double cover $\tilde{M}$.

**Group Theory:** $\pi_1(\tilde{M}) < \pi_1(M)$ of index 2.

**Prime decomposition:**

**Topology:** Cut orientable $M$ along a maximal system of $S^2$’s; glue $D^3$ into boundary of each piece to decompose $M$ as $M = M_1 \# M_2 \# \cdots \# M_k$, there $M_i$ is prime.

**Group Theory:** $\pi_1(M) = \pi_1(M_1) \ast \pi_1(M_2) \ast \cdots \pi_1(M_k)$. 

Thm. (BHJ 2016) Autostackable groups are closed under free/direct/graph products.
Decomposing $M$: Part I

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**Group Theory:** $\pi_1(\tilde{M}) < \pi_1(M)$ of index 2.

**Thm.** (Brittenham, Hermiller, Johnson 2016)
*Autostackable groups are closed under finite index supergroups.*

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Decomposing $M$: Part II

**Topology:** Cut $M_i$ along a maximal system of embedded, incompressible tori, to obtain 3-manifolds $M_{i,j}$ with incompressible torus boundary.

**Group Theory:** Use Seifert-vanKampen, $\pi_1(M_i)$ decomposes as the fundamental group of a graph of groups. vertex groups: $\pi_1(M_{i,j})$, edge groups: $\mathbb{Z}^2$
Decomposing $M$: Part II

**JSJ decomposition:**

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- **vertex groups:** $\pi_1(M_{i,j})$
- **edge groups:** $\mathbb{Z}^2$

**Geometrization:**

**Topology:** A prime 3-manifold $M_i$ is either:

- Geometric: that is admits, a complete metric based one of Thurston’s 8 geometries;
- Non-geometric: admits a non-trivial JSJ decomposition with each $M_{i,j}$ either: ● Seifert fibered or ● hyperbolic

**Thm** (BHH, 2014): If $M^3$ is closed and geometric, then $\pi_1(M^3)$ is autostackable
The class of autostackable groups is NOT closed under taking graphs of groups!

**Example (Mihalova; Hermiller–Martinez-Perez):**
There is a group obtained by taking an HNN-extension of $F_2 \times F_2$ which has unsolvable word problem. It is stackable but cannot be algorithmically stackable.

**We need a stronger condition on** $\pi_1(M_{i,j})$. **They must be autostackable respecting the peripheral subgroups.**
Closure for fundamental groups of graphs of groups

**Defn.** $G$ is **autostackable respecting** a subgroup $H$ if there are finite generating sets $G = \langle A \rangle$, and $A$ contains generators, $B$, for $H$, such that $G$ is autostackable over $A$ with a spanning tree $T$ satisfying:

- **Subgroup closure:** $T$ contains a spanning tree $T'$ for $H$ over $B$, and for all edges $e \in \Gamma(H, B)$, $\Phi(e) \subseteq \Gamma(H, B)$.
- **$H$–translation invariance:** $T \setminus T'$ is an $H$–orbit of a transversal tree for $H$ in $G$.
- for all $h \in H$, and edges $e \not\in \Gamma(H, B)$, $\Phi(h \cdot e) = h \cdot \Phi(e)$. 


Closure for fundamental groups of graphs of groups
Closure for fundamental groups of graphs of groups

\[ H \rightarrow T' \rightarrow T'' \]
Closure for fundamental groups of graphs of groups
Closure for fundamental groups of graphs of groups
Thm. (BHS, 2016)
Let \( \Lambda \) be a finite connected graph with vertex groups \( G_v \) and edge groups \( G_e \) such that for each endpoint \( v \) of an edge \( e \), \( G_v \) is autostackable respecting \( G_e \). The fundamental group of this graph of groups is autostackable.
Hyperbolic pieces

**Thm. (BHS 2016)** If $G$ is finitely generated and hyperbolic relative to a collection $\mathcal{H}$ of abelian subgroups, then for all $H \in \mathcal{H}$, the group $G$ is autostackable respecting $H$.

**Ingredients:**

**Thm. (BHS, 2016)** Let $G$ be a finitely generated group, and $H$ a finitely generated autostackable subgroup. Suppose that the pair $(G, H)$ is strongly coset automatic, then $G$ is autostackable respecting $H$.

\[ u, v \in \text{transversal language}, \, ux = hv, \, h \in H. \]
Seifert fibered pieces

**Thm.** (BHS, 2016) If $M$ is a Seifert-fibered 3-manifold with incompressible torus boundary, then for each $T^2$ boundary component, $\pi_1(M)$ is autostackable respecting $\pi_1(T^2)$.

**Ingredients:**

**Topology:** $M$ is a circle bundle over a 2–dimensional orbifold.
Seifert fibered pieces

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**Ingredients:**

**Group Theory:** (Seifert 1933) Let $M$ be a Seifert fibered 3–manifold and let $X$ be the base orbifold of the fibration. Then, there is a short exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow \pi^o_1(X) \rightarrow 1$$

where $\pi^o_1(X)$ is the orbifold fundamental group of $X$. 
**Seifert fibered pieces**

**Thm. (BHS, 2016)** If $M$ is a Seifert-fibered 3-manifold with incompressible torus boundary, then for each $T^2$ boundary component, $\pi_1(M)$ is autostackable respecting $\pi_1(T^2)$.

**Ingredients:**

\[ 1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow \pi_1^0(X) \rightarrow 1 \]

**Topology:** Cases: $X$ hyperbolic or Euclidean

**Group Theory:** Cases: $\pi_1^0(X)$ is either Gromov hyperbolic or virtually $\mathbb{Z}$.

**Thm. (BHS, 2016)** Autostackability respecting subgroups behaves nicely under extensions.
Open questions

• Are the normal forms for the autostackable structures on closed 3-manifolds quasi-geodesics?
  **Partial Answer:** True for each piece of the JSJ decomposition

• What bounds can be found for the time/space complexity of the word problem solution from the autostackable structures for closed 3-manifolds?

• Find a procedure that upon input of a finite presentation of a closed 3-manifold group and a well-founded ordering, can output the associated autostackable structure (if one exists for the given presentation/ordering).

• Is there a finitely presented group that is not stackable?
  ( Conj. (Tschantz): Yes; a group that is not tame combable.)
Thank You

Thank You!