AN INVERSE COEFFICIENT PROBLEM FOR AN INTEGRO-DIFFERENTIAL EQUATION

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Abstract. In this paper we consider the inverse coefficient problem of recovering the function $\phi(x)$ system of partial differential equations that can be reduced to a second order integro-differential equation $-u_{xx}+c(x)u_x+d\phi(x)u-\gamma d\phi(x)\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau=0$ with boundary conditions. We prove the existence and uniqueness of solutions to the inverse problem and develop a numerical algorithm for solving this problem. Computational results for some examples are presented.

1. Introduction

The direct problem we study is a system of partial differential equations of the form

$$c_1(x)u_x + a_t = D_1 u_{xx} \tag{1.1}$$

$$a_t = \gamma(\phi(x)u - a) \tag{1.2}$$

for $0 \le x \le L,\ 0 \le t \le T$ subject to the boundary and initial conditions

$$u(0,t) = \mu(t),\ 0 \le t \le T \tag{1.3}$$

$$u(L,t) + \beta u_x(L,t) = 0,\ 0 \le t \le T \tag{1.4}$$

$$a(x,0) = 0,\ 0 \le x \le L. \tag{1.5}$$

It is possible to consider the direct problem (1.1)–(1.5) as a one dimensional mathematical model of adsorption dynamics. Here the function $\phi(x)$ describes a spatially varying property of the adsorbing medium. The inverse problem (IP) that we consider here is as follows: given that $u(x,t)$ is a solution to the direct problem given by (1.1)–(1.5), and exact data

$$g(x) = u(x,T),\ 0 \le x \le L, \tag{1.6}$$

to determine the coefficient function $\phi(x)$, $x \in [0,L]$, given in (1.2).

We shall reduce the direct problem (1.1)–(1.5) to a boundary value problem for an integro-differential equation, and consider the inverse problem for this equation. We first derive the properties of solutions to this integro-differential equation that are needed for an analysis of the inverse problem. The main results of this paper are theorems of existence and uniqueness of solutions to the inverse problem. We shall use the monotone operator method [4] to prove existence of solutions to the inverse problem. Other examples of the application of this method to solve inverse problems can be found in the papers [1],[2] and [3]. In the last section of this

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paper we develop a numerical method for solving the inverse problem and illustrate this method with several numerical examples.

2. An Integro-Differential Equation

2.1. **Classical Boundary Value Problems.**

We first consider the simple boundary value problem

$$Lv = -v'' + p(x)v' + q(x)v = H(x), \ 0 \le x \le L \tag{2.1}$$

$$v(0) = 0 \tag{2.2}$$

$$v(L) + \beta v_x(L) = 0. \tag{2.3}$$

We assume that β and $q(x)$ are positive and the functions $p(x), q(x), H(x)$ are continuous. The boundary value problem (2.1)–(2.3) can be put into selfadjoint form by multiplying (2.1) by an integrating factor $e^{-\int_0^x p(\xi)d\xi}$ to obtain

$$-(e^{-\int_0^x p(\xi)d\xi}v')' + e^{-\int_0^x p(\xi)d\xi}q(x)v(x) = e^{-\int_0^x p(\xi)d\xi}H(x). \tag{2.4}$$

The Green's function $G(x,\xi)$ for (2.4), (2.2) and (2.3) is easily seen to be non-negative over its domain. First of all we note that equations (2.1) and (2.4) for the homogeneous case $H(x) = 0$ are equivalent. Next, we choose a solution $y(x)$ to the homogeneous differential equation $Ly = 0$ such that $y(0) = 0$ and $y'(0) = 1$. There can be no positive local maxima in the interior of the interval $[0, L]$, since at such an x_0 we have $y''(x_0) = q(x_0)y(x_0) > 0$. Consequently, y is positive and increasing on the interval $(0, L]$. A similar argument yields a solution $z(x)$ to the homogeneous equation $Ly = 0$ satisfying the right boundary conditions $z(L) = \beta$ and $z'(L) = -1$ which is positive and decreasing on the interval $[0, L]$. Recall that the Green's function for the boundary value problem (2.4), (2.2) and (2.3) is given by

$$G(x,\xi) = \begin{cases} -\frac{y(\xi)z(x)}{C}, & x \ge \xi \\ -\frac{y(x)z(\xi)}{C}, & x < \xi \end{cases}$$

$$C = e^{-\int_0^x p(s)ds}\{y(x)z'(x) - y'(x)z(x)\} = -z(0).$$

Consequently, $G(x,\xi)$ is also non-negative. Moreover, the solution to (2.1)–(2.3) is given by

$$v(x) = \int_0^L K(x,\xi)H(\xi)\,d\xi, \tag{2.5}$$

where $K(x,\xi) = G(x,\xi)e^{-\int_0^\xi p(s)ds}$, which is also a non-negative function.

If we set $k(x) = z(x)/z(0)$, we see that $k(x)$ is a positive decreasing solution to (2.1) satisfying the right boundary condition $k(L) + \beta k'(L) = 0$ and left boundary condition $k(0) = 1$. This function is important for our analysis of the direct problem. In particular, it occurs in the solution bounds that appear in Theorem 2.3 below, as well as in our analysis of the inverse problem in Section 4. It also simplifies the description of the Green's function for the non-selfadjoint problem to

$$K(x,\xi) = \begin{cases} e^{-\int_0^\xi p(s)ds}y(\xi)k(x), & x \ge \xi \\ e^{-\int_0^\xi p(s)ds}y(x)k(\xi), & x < \xi \end{cases} \tag{2.6}$$

To emphasize the dependence on a parameter such as $q(x)$, we may write $k(x) = k(x;q)$, $y(x) = y(x;q)$ and $z(x) = z(x;q)$.

The following fact will be used in our analysis of the inverse problem.

Lemma 2.1. *If $0 < q(x) \le C_0$ for constant C_0 and $x \in [0,L]$, then the families of functions $K_x(x,\xi;q)$, $k'(x;q)$ are uniformly bounded by a constant $\widehat{C}_0$ independent of $q(x)$.*

Proof. In view of (2.6) it is clearly sufficient to show that $y'(x)$ and $k'(x)$ are uniformly bounded on the interval $[0,L]$. Both of these functions are the second coordinate of solutions to an initial value problem

$$\begin{aligned} V' &= \begin{bmatrix} 0 & 1 \\ q(x) & p(x) \end{bmatrix} V = AV \\ V(a) &= V_0. \end{aligned}$$

In the case of $y(x;q)$, $a = 0$ and $V_0^T = (0,1)$. In the case of $k(x;q)$, $a = L$ and $V_0^T = (\beta/z(0), -1/z(0))$. In the latter case, note that since $z(x;q)$ is decreasing, $z(0;q) \ge \beta$, so that $\|V_0\| \le 1 + 1/\beta$. In either case, $\|V_0\|$ is bounded by the constant $1 + 1/\beta$, which is independent of $q(x)$. To establish a bound for $y(x;q)$, integrate the differential equation from 0 to x and take uniform norms to obtain that

$$\|V(x)\| \le \|V_0\| + \int_0^x \|A(s)\| \, \|V(s)\| \, ds \le (1 + \frac{1}{\beta}) + \int_0^x (\|p\| + C_0 + 1) \, \|V(s)\| \, ds.$$

It follows from Gronwall's inequality that for $x \in [0,L]$,

$$\|V(x)\| \le (1 + \frac{1}{\beta}) \exp \int_0^L (\|p\| + C_0 + 1) \, ds \equiv \widehat{C}_0$$

A similar argument gives the same bound for the case of $k(x;q)$, which proves the lemma. $\square$

2.2. **Integro-Differential Boundary Value Problems.** We next consider a integro-differential equation which includes the reformulation of the direct problem in Section 3 for the function $u = u(x,t)$. Let $Q_T = [0,L] \times [0,T]$.

Theorem 2.2. *Let β, γ and $q(x)$ be positive, $F(x,t)$ and $\mu(t)$ non-negative and all functions continuous in the boundary value problem*

$$-u_{xx} + p(x)u_x + q(x)u - \gamma q(x) \int_0^t e^{-\gamma(t-\tau)} u(x,\tau) \, d\tau = F(x,t), \; (x,t) \in Q_T \tag{2.7}$$

$$u(0,t) = \mu(t), \; 0 \le t \le T \tag{2.8}$$

$$u(L,t) + \beta u_x(L,t) = 0, \; 0 \le t \le T. \tag{2.9}$$

Then this problem has a unique non-negative solution $u(x,t) \in C[Q_T]$ with continuous partials $u_x(x,t)$ and $u_{xx}(x,t)$. Moreover, if $F_t(x,t)$ and $\mu'(t)$ are continuous, then so is $u_t(x,t)$.

Proof. Let $u(x,t)$ be a solution to the boundary value problem (2.7)–(2.9). We introduce the function $v(x,t) = u(x,t) - \mu(t)k(x)$, where $k(x)$ is as in the discussion preceding this theorem.

Then the function $v(x,t)$ solves the boundary value problem

$$-v_{xx} + p(x)v_x + q(x)v = F(x,t) + \gamma q(x)\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau, (x,t) \in Q_T \tag{2.10}$$

$$v(0,t) = 0,\ 0 \le t \le T \tag{2.11}$$

$$v(L,t) + \beta v_x(L,t) = 0,\ 0 \le t \le T. \tag{2.12}$$

Apply (2.5) to $v(x,t)$ and replace $v(x,t)$ by $u(x,t) - \mu(t)k(x)$ to obtain that the boundary value problem (2.7)–(2.9) is equivalent to the integral equation

$$u(x,t) = \mu(t)k(x) + \int_0^L K(x,\xi)F(\xi,t)\,d\xi + \int_0^L K(x,\xi)\gamma q(\xi)\int_0^t e^{-\gamma(t-\tau)}u(\xi,\tau)\,d\tau\,d\xi,$$

that is,

$$u(x,t) = f(x,t) + \int_0^t\int_0^L e^{-\gamma(t-\tau)}K(x,\xi)\gamma q(\xi)u(\xi,\tau)\,d\xi\,d\tau, \tag{2.13}$$

where $f(x,t) = \mu(t)k(x) + \int_0^L K(x,\xi)F(\xi,t)\,d\xi$. By the non-negativity of $F(x,t)$, $\mu(t)$, $k(x)$ and the kernel $K(x,\xi)$, we see that $f(x,t)$ is a non-negative function. The double integration in (2.13) yields an integral equation of Volterra type for the function $u(x,t)$. Moreover, the integral operator P given by

$$Pu = \int_0^t\int_0^L e^{-\gamma(t-\tau)}K(x,\xi)\gamma q(\xi)u(\xi,\tau)\,d\xi\,d\tau$$

has a continuous non-negative kernel. It follows that the unique solution to the integral equation (2.13) is given by the Neumann series

$$u = f + Pf + P^2f + \cdots$$

which shows that $u(x,t)$ is non-negative. Finally, observe that if $F(x,t)$ and $\mu(t)$ have continuous derivatives $F_t(x,t)$ and $\mu'(t)$, then we obtain from (2.13) that $u(x,t)$ has continuous derivative $u_t(x,t)$. □

We next establish bounds for solutions in the source-free case. In the following result, we use the function $k(x)$ used in the construction of the Green's function of (2.6). Recall that $k(x) > 0$, for $0 \le x \le L$.

Theorem 2.3. *Let β, γ and $q(x)$ be positive, $F(x,t) = 0$, $\mu(t)$ non-negative and all functions continuous in the boundary value problem (2.7)–(2.9). Then for $(x,t) \in Q_T$ the solution $u(x,t)$ to the this boundary value problem satisfies*

$$\mu(t)k(x) \le u(x,t) \le \max_{0\le t\le T}\mu(t).$$

Proof. Suppose that for some $(x_0,t_0) \in Q_T$,

$$u(x_0,t_0) = \max_{(x,t)\in Q_T} u(x,t).$$

The case $u(x_0,t_0)=0$ is trivial, so suppose $u(x_0,t_0)>0$. From (2.9) we have that $x_0<L$. If $x_0>0$, then the integro-differential equation at (x_0,t_0) yields

$$\begin{aligned} 0 &= u_{xx}(x_0,t_0) - q(x_0)u(x_0,t_0) + \gamma q(x_0)\int_0^{t_0} e^{-\gamma(t_0-\tau)}u(x_0,\tau)\,d\tau \\ &< u_{xx}(x_0,t_0) - q(x_0)u(x_0,t_0)\left(1-\gamma\int_0^{t_0} e^{-\gamma(t_0-\tau)}\,d\tau\right) < 0, \end{aligned}$$

a contradiction. This proves that $x_0=0$ and $u(x,t)\le \mu(t_0)$, which establishes the upper bound on $u(x,t)$.

For the lower bound, observe that $v(x,t)=u(x,t)-\mu(t)k(x)$ satisfies the boundary value problem given by the integro-differential equation

$$-v_{xx}+p(x)v_x+q(x)v-\gamma q(x)\int_0^t e^{-\gamma(t-\tau)}v(x,\tau)\,d\tau = \gamma q(x)\int_0^t e^{-\gamma(t-\tau)}\mu(\tau)k(x)\,d\tau$$

together with boundary conditions (2.11)–(2.12). Since the right hand side of the integro-differential equation is non-negative, Theorem 2.2 may be applied to yield $v(x,t)\ge 0$, which establishes the lower bound on $u(x,t)$. □

We note that a sharper upper bound can be established with the additional hypothesis that $\mu'(t)\ge 0$, namely that $u(x,t)\le\mu(t)$ for $(x,t)\in Q_T$. We omit the details.

3. The Direct Problem

We can give a fairly complete analysis of the direct problem (1.1)-(1.5), subject to suitable conditions on the parameters of the problem. We shall make the following assumptions on the parameters of the problem throughout this section.

Parameter Conditions:

(1) $\phi\in C[0,L]$ and $\phi(x)>0$ for $x\in[0,L]$.
(2) $c_1(x)\in C[0,L]$.
(3) The constants D_1, γ and β are positive.
(4) $\mu\in C[0,L]$ and μ is non-negative on $[0,T]$.
(5) $\mu\in C^1[0,L]$, $\mu(0)=0$ and μ' is positive on $(0,T]$.
(6) $\mu'(t)-\gamma\int_0^t e^{-\gamma(t-\tau)}\mu'(\tau)\,d\tau\ge 0$, for $0\le t\le T$.

We note that condition 6 is satisfied if, for example, $\mu'(0)\ge 0$ and $\mu''(t)\ge 0$. By a *solution* to the direct problem (1.1)–(1.5), we mean a pair of functions $u,a\in C[Q_T]$ such that $u_x,u_{xx},a_t\in C[Q_T]$ and u,a solve the equations (1.1)–(1.5). In order to establish the existence and uniqueness of a solution to the direct problem, we only use parameter conditions 1-4. However, in order to provide bounds for solutions and a comparison theorem, we also require parameter conditions (5) and (6).

Theorem 3.1. *Assume parameter conditions 1-4. Then the direct problem has a unique solution $u(x,t)$, $a(x,t)$ with both functions non-negative.*

Proof. We first recast the direct problem as follows. Integrate (1.2) and use the initial condition (1.5) to obtain that for all $(x,t) \in Q_T$,

$$a(x,t) = \gamma\phi(x)\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau. \tag{3.1}$$

If we substitute (1.2) and (3.1) into (1.1), we obtain the equation

$$c_1(x)u_x + \gamma\phi(x)u - \gamma^2\phi(x)\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau = D_1 u_{xx}.$$

Now let $c(x) = c_1(x)/D_1$ and $d = \gamma/D_1$, and this equation becomes

$$-u_{xx} + c(x)u_x + d\phi(x)u - \gamma d\phi(x)\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau = 0. \tag{3.2}$$

It follows that the direct problem is equivalent to the problem (3.1), (3.2), (1.3) and (1.4). From Theorem 2.2 we have that there exists a unique non-negative solution $u(x,t)$ to the problem (3.2), (1.3) and (1.4) with continuous derivatives u_x, u_{xx}. If we define $a(x,t)$ using (3.1), we obtain that a and a_t are continuous in Q_T and a is non-negative. Hence, $u(x,t), a(x,t)$ is the unique, non-negative solution to the direct problem. $\square$

The bound established in Theorem 2.3 has the following application to the direct problem.

Theorem 3.2. *Assume parameter conditions 1-5. If u, a is a solution to the direct problem, then $u_t \in C[Q_T]$ and for $(x,t) \in Q_T$*

$$\mu'(t)k(x) \le u_t(x,t) \le \max_{0\le t\le T}\mu'(t),$$

where $k(x)$ is the solution to (2.1) for $p(x) = c(x)$, $q(x) = d\phi(x)$, $H(x) = 0$ satisfying boundary conditions $k(0) = 1$ and $k(L) + \beta k'(L) = 0$.

Proof. From Theorem 2.2 we have $u_t \in C[Q_T]$. Observe that for $(x,t) \in Q_T$

$$\frac{\partial}{\partial t}\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau = u(x,t) - \gamma\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau. \tag{3.3}$$

An application of (3.3) and integration by parts show that

$$\frac{\partial}{\partial t}\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau = e^{-\gamma t}u(x,0) + \int_0^t e^{-\gamma(t-\tau)}u_t(x,\tau)\,d\tau.$$

Since $\mu(0) = 0$ we have that $u(x,0) = 0$. Therefore,

$$\frac{\partial}{\partial t}\int_0^t e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau = \int_0^t e^{-\gamma(t-\tau)}u_t(x,\tau)\,d\tau.$$

Now set $v(x,t) = u_t(x,t)$, differentiate (3.2), (1.3) and (1.4) with respect to t, and we obtain that for $(x,t) \in Q_T$

$$-v_{xx} + c(x)v_x + d\phi(x)v - \gamma d\phi(x)\int_0^t e^{-\gamma(t-\tau)}v(x,\tau)\,d\tau = 0 \tag{3.4}$$

$$v(0,t) = \mu'(t) \tag{3.5}$$

$$v(L,t) + \beta v_x(L,t) = 0. \tag{3.6}$$

Theorem 2.2 implies that $v(x,t) = u_t(x,t) \geq 0$ for $(x,t) \in Q_T$. Theorem 2.3 may be applied to $v(x,t)$ to yield that $\mu'(t)k(x) \leq v(x,t) = u_t(x,t) \leq \max_{0\leq t\leq T} \mu'(t)$ for $(x,t) \in Q_T$. $\square$

Finally, the additional parameter condition 6 is sufficient to obtain a comparison theorem. To emphasize the dependence of the solution of the direct problem on the function ϕ, we write $u(x,t) = u(x,t;\phi)$, $a(x,t) = a(x,t;\phi)$.

Theorem 3.3. *Assume parameter conditions 1-6. Let $u(x,t;\phi_i), a(x,t;\phi_i)$ be a solution to the direct problem corresponding to ϕ_i, $i = 1, 2$. If $\phi_1(x) \leq \phi_2(x)$ for $x \in [0, L]$, then $u_t(x,t;\phi_1) \geq u_t(x,t;\phi_2)$ for $(x,t) \in Q_T$.*

Proof. Just as in the proof of Theorem 3.2, we have that $v(x,t) = u_t(x,t;\phi)$ satisfies the boundary value problem (3.4)–(3.6). Let $v_i(x,t) = u_t(x,t;\phi_i)$, $i = 1, 2$. Also define $V_i = \int_0^t e^{-\gamma(t-\tau)} v_i(x,\tau)\, d\tau$, $i = 1, 2$, $w = v_1 - v_2$ and $W = V_1 - V_2$. It follows from (3.3) that the governing differential equation may be written in the forms

$$-(v_i)_{xx} + c(x)(v_i)_x + d\phi_i(x)(V_i)_t = 0.$$

Subtract these equations for $v_1(x,t)$ and $v_2(x,t)$ and rearrange terms to obtain

$$-w_{xx} + c(x)w_x + d\phi_1(x)W_t = d(\phi_2(x) - \phi_1(x))(V_2)_t.$$

It therefore suffices to show that $(V_2)_t$ is non-negative, since an application of Theorem 2.2 then shows that $w(x,t) \geq 0$, as required.

Since $v_2(x,t)$ is a solution to (3.4) with $\phi(x) = \phi_2(x)$ and boundary conditions (3.5)–(3.6), it is easily seen that V_2 is also a solution to (3.4) with $\phi = \phi_2$ and the conditions $V_2(0,t) = \int_0^t e^{-\gamma(t-\tau)} \mu'(\tau)\, d\tau$ and $V_2(L,t) + \beta(V_2)_x(L,t) = 0$. Observe that

$$(V_2)_t(0,t) = \mu'(t) - \gamma \int_0^t e^{-\gamma(t-\tau)} \mu'(\tau)\, d\tau \geq 0,$$

where the last inequality follows from parameter condition 6. It follows from Theorem 3.2 that $(V_2)_t(x,t) \geq 0$, which completes the proof. $\square$

4. The Inverse Problem

4.1. **An Operator Formulation of the Problem.** Given data $g(x)$ $x \in [0, L]$, we define a *solution* to the inverse problem (IP), as given in Section 1, to be a positive continuous function $\phi(x)$, $x \in [0, L]$, such that the direct problem (1.1)-(1.5) has solution $u(x,t)$ satisfying (1.6). We first note that the data $g(x)$ of (1.6) must also satisfy the boundary conditions $g(0) = \mu(T)$ and $g(L) + \beta g'(L) = 0$, since $u(x,T)$ satisfies these conditions. Moreover, it is clear that $g(x)$ must have a continuous second derivative.

The inverse problem can be expressed in terms of an operator equation as follows. Set $t = T$ in (3.2) and use (1.6) to obtain

$$\begin{aligned} g''(x) - c(x)g'(x) &= d\phi(x)u(x,T) - \gamma d\phi(x) \int_0^T e^{-\gamma(T-\tau)} u(x,\tau)\, d\tau \\ &= d\phi(x) \int_0^T e^{-\gamma(T-\tau)} u_t(x,\tau)\, d\tau, \end{aligned}$$

or

$$\phi(x) = \frac{g''(x) - c(x)g'(x)}{d\int_0^T e^{-\gamma(T-\tau)} u_t(x,\tau)\, d\tau}.$$

We see from this equation that the data function $g(x)$ should satisfy $g''(x) - c(x)g'(x) > 0$ since $\phi(x)$ and $u_t(x,t)$ are positive, To summarize, the properties that a data function $g(x)$ must have are:

Data Conditions:

(1) $g(x) \in C^2[0, L]$ and $g(x)$ is a positive function.
(2) $g(0) = \mu(T)$ and $g(L) + \beta g'(L) = 0$.
(3) $g''(x) - c(x)g'(x) \in C[0, L]$ and is a positive function on $[0, L]$.

We assume the function $g(x)$ satisfies data conditions 1-3 and define the operator A by

$$A\phi = \frac{g''(x) - c(x)g'(x)}{d\int_0^T e^{-\gamma(T-\tau)} u_t(x,\tau;\phi)\, d\tau}, \tag{4.1}$$

where $u(x,t;\phi)$ is the solution of the direct problem (1.1)-(1.5) for the given $\phi(x)$. It follows that if the function $\phi(x)$ is a solution to the inverse problem, then

$$\phi = A\phi. \tag{4.2}$$

Conversely, suppose that $\widetilde{\phi}$ is a positive continuous solution to (4.2). Then

$$\begin{aligned} g''(x) - c(x)g'(x) &= d\widetilde{\phi}(x)u(x,T;\widetilde{\phi}) - d\widetilde{\phi}(x)\gamma\int_0^T e^{-\gamma(T-\tau)} u(x,\tau;\widetilde{\phi})\, d\tau \\ &= u_{xx}(x,T;\widetilde{\phi}) - c(x)u_x(x,T;\widetilde{\phi}). \end{aligned}$$

Since both $g(x)$ and $u(x,T;\widetilde{\phi})$ satisfy the same boundary conditions, it follows that $g(x) = u(x,T;\widetilde{\phi})$ so that $\widetilde{\phi}(x)$ is a solution to the inverse problem.

4.2. **Properties of the Operator A.** Throughout this section, we shall assume that parameter conditions 2-6 hold. We now examine some properties of the operator A. It follows from Theorem 3.2 that for a positive $\phi(x)$,

$$A\phi = \frac{g''(x) - c(x)g'(x)}{d\int_0^T e^{-\gamma(T-\tau)} u_t(x,\tau;\phi)\, d\tau} \geq \frac{g''(x) - c(x)g'(x)}{d\int_0^T e^{-\gamma(T-\tau)} \mu_M\, d\tau} \equiv h(x) \tag{4.3}$$

where $\mu_M = \max_{0\leq t\leq T} \mu'(t)$. We define

$$E = \{\phi(x) \in C[0, L] \,|\, \phi(x) \geq h(x),\, x \in [0, L]\}.$$

Then $E \subset P$, the set of positive functions in $C[0, L]$. Throughout the following, we use the uniform norms $\|\cdot\|$ in $C[0, L]$ and $C(Q_T)$. The operator A is said to be *monotone* (see, for example, [4, p. 581]) if, for functions $\phi(x)$ and $\Psi(x)$ in the domain of A with $\phi(x) \leq \Psi(x)$ for $x \in [0, L]$, we have $A\phi(x) \leq A\Psi(x)$ for $x \in [0, L]$. The next two theorems exhibit the essential properties of the operator A.

Theorem 4.1. *The operator $A : P \to E$ given by (4.1) is a continuous and monotone operator.*

Proof. First observe that by the inequality (4.3) the operator A does indeed map positive elements of $C[0,L]$ into the set E. Define

$$B\phi = \int_0^T e^{-\gamma(T-\tau)} u_t(x,\tau;\phi)\,d\tau, \tag{4.4}$$

so that

$$A\phi = \frac{g'' - cg'}{d}\left(\frac{1}{B\phi}\right).$$

Now $A\phi$ is bounded and bounded away from zero since by Theorem 3.2,

$$0 < k(x;\phi)\int_0^T e^{-\gamma(T-\tau)}\mu'(\tau)\,d\tau \le B\phi(x) \le \int_0^T e^{-\gamma(T-\tau)}\mu_M\,d\tau, \tag{4.5}$$

where the first inequality is strict by parameter condition 5. Therefore A is continuous if B is continuous. Moreover, since the integral operator $I\psi = \int_0^T e^{-\gamma(T-\tau)}\psi(x,\tau)\,d\tau$ is clearly continuous in the uniform norm, it suffices to show that the operator $C\phi = u_t(x,T;\phi)$ is continuous.

Let $v_i(x,t) = u_t(x,t;\phi_i)$, $i = 1,2$, and let $w(x,t) = v_1(x,t) - v_2(x,t)$. Each v_i solves the boundary problem (3.4)-(3.6) with $\phi = \phi_i$. Subtract these differential equations and rearrange terms to obtain

$$-w_{xx} + c(x)w_x + d\phi_1(x)w - \gamma d\phi_1(x)\int_0^t e^{-\gamma(t-\tau)}w(x,\tau)\,d\tau = F(x,t),$$

where

$$F(x,t) = \gamma d(\phi_1(x) - \phi_2(x))\int_0^t e^{-\gamma(t-\tau)}v_2(x,\tau)\,d\tau - d(\phi_1(x)-\phi_2(x))v_2(x,t). \tag{4.6}$$

Furthermore, $w(x,t)$ satisfies homogeneous boundary conditions (3.5) and (3.6). As in Section 2, we obtain that $w(x,t)$ satisfies an integral equation of the form

$$w(x,t) = \int_0^L K(x,\xi;\phi_1)F(\xi,t)\,d\xi + \int_0^t\int_0^L K(x,\xi;\phi_1)\gamma d\phi_1(\xi)e^{-\gamma(t-\tau)}w(\xi,\tau)\,d\xi\,d\tau, \tag{4.7}$$

where $K(x,\xi;\phi_1)$ is a continuous function on the square $0 \le x,\xi \le L$. Let $K_M = \max_{0\le x,\xi\le L}|K(x,\xi;\phi_1)|$, $\mu_M = \max_{0\le t\le T}\mu'(t)$ and $W(t) = \|w(\cdot,t)\|$ so that we obtain from (4.6), (4.7) and Theorem 3.2 that

$$W(t) \le LK_M d\mu_M(\gamma T+1)\,\|\phi_1 - \phi_2\| + LK_M\gamma d\,\|\phi_1\|\int_0^t W(\tau)\,d\tau.$$

It follows from Gronwall's inequality that

$$W(t) \le LK_M d\mu_M(\gamma T+1)\,\|\phi_1-\phi_2\|\,e^{LK_M\gamma d\|\phi_1\|t},$$

from which the continuity of B and therefore A, follows.

To establish the monotonicity of A, apply Theorem 3.3 to obtain that if $\phi_1(x) \le \phi_2(x)$ then $u_t(x,t;\phi_1) \ge u_t(x,t;\phi_2)$. Therefore,

$$\int_0^T e^{-\gamma(T-\tau)}u_t(x,\tau;\phi_2)\,d\tau \le \int_0^T e^{-\gamma(T-\tau)}u_t(x,\tau;\phi_1)\,d\tau. \tag{4.8}$$

Both terms are positive by (4.5). Take reciprocals, multiply by the positive factor $(g''(x) - c(x)g'(x))/d$ and monotonicity follows, which completes the proof. □

Theorem 4.2. *The operator $A : P \to E$ given by (4.1) maps bounded sets into equicontinuous sets.*

Proof. We consider a bounded set of functions

$$P_0 = \{\phi(x) \in C[0, L] \,|\, 0 < \phi(x) \le \phi_0(x),\ x \in [0, L]\}$$

where $\phi_0(x)$ is a continuous function. Let $0 \le x_1, x_2 \le L$. Define the operator B as in (4.4) and define $r(x) = (g''(x) - c(x)g'(x))/d$, so that $A\phi = r/B\phi$. It follows that

$$A\phi(x_1) - A\phi(x_2) = \frac{(r(x_1) - r(x_2))B\phi(x_2) + r(x_2)(B\phi(x_2) - B\phi(x_1))}{B\phi(x_1)B\phi(x_2)}.$$

There is a constant $C_1 > 0$ independent of x and ϕ such that $B\phi(x_1)B\phi(x_2) > C_1$ and a constant $C_2 > 0$ such that B$\phi(x) < C_2$ thanks to (4.5). Let $\omega_r(h)$ be the modulus of continuity of $r(x)$ and we obtain that

$$|A\phi(x_1) - A\phi(x_2)| \le \frac{\omega_r(|x_1 - x_2|)C_2 + \|r\|\,|B\phi(x_2) - B\phi(x_1)|}{C_1}. \tag{4.9}$$

We shall show that for any $\phi \in P_0$ there exists a constant C_3 such that

$$|B\phi(x_2) - B\phi(x_1)| \le C_3\,|x_2 - x_1|\,.$$

This fact, together with (4.9), establishes the theorem.

It follows from the proofs of Theorem 3.2 and Theorem 2.2 that $v(x, t) = u_t(x, t; \phi)$ satisfies the integral equation

$$v(x, t) = f(x, t) + \int_0^t \int_0^L e^{-\gamma(t-\tau)} K(x, \xi; \phi)\gamma d\phi(\xi)v(\xi, \tau)\, d\xi\, d\tau, \tag{4.10}$$

where $f(x, t) = \mu'(t)k(x)$. Evaluate (4.10) at $x = x_1, x_2$ and subtract to obtain

$$\begin{aligned} v(x_1, t) - v(x_2, t) &= \mu'(t)(k(x_1) - k(x_2)) + \\ &\int_0^t \int_0^L e^{-\gamma(t-\tau)}(K(x_1, \xi; \phi) - K(x_2, \xi; \phi))\gamma d\phi(\xi)v(\xi, \tau)\, d\xi\, d\tau. \end{aligned} \tag{4.11}$$

If ξ is between x_1 and x_2, we may write

$$K(x_1, \xi; \phi) - K(x_2, \xi; \phi) = K(x_1, \xi; \phi) - K(\xi, \xi; \phi) + K(\xi, \xi; \phi) - K(x_2, \xi; \phi).$$

Thus we can apply the mean value theorem to each term, as well as to the term $k(x_1) - k(x_2)$. Lemma 2.1 yields a uniform bound $\widehat{C}_0$ on the functions $K_x(x, \xi; \phi)$ and $k'(x)$ independent of ϕ. Let $\mu_M = \max_{t\in[0,T]} \mu'(t)$ and $C = \|\phi_0\|$ Take absolute values of (4.11) and we obtain

$$\begin{aligned} |v(x_1, t) - v(x_2, t)| &\le \mu_M\widehat{C}_0\,|x_1 - x_2| + \int_0^t 2\widehat{C}_0\,|x_1 - x_2|\,\gamma dC\mu_M L\, d\tau \\ &\le \widehat{C}_0\mu_M\,(1 + 2\gamma dCLT)\,|x_1 - x_2|\,. \end{aligned}$$

It follows that

$$\begin{aligned} |B\phi(x_1) - B\phi(x_2)| &\le \int_0^T e^{-\gamma(T-\tau)}\,|v(x_1, t) - v(x_2, t)|\, d\tau \\ &\le T\widehat{C}_0\mu_M\,(1 + 2\gamma dCLT)\,|x_1 - x_2|\,, \end{aligned}$$

which completes the proof. □

4.3. **Properties of Solutions to the Inverse Problem.**

Theorem 4.3. *If parameter conditions 2-6 and data conditions 1-3 are satisfied, then a necessary and sufficient condition for the inverse problem (IP) to have a solution is that there exist a positive function $\phi_0(x) \in C[0, L]$ for which $A\phi_0(x) \leq \phi_0(x)$, $0 \leq x \leq L$.*

Proof. If (IP) has a solution $\phi(x)$, we have already in (4.2) that $A\phi = \phi$ and we may take $\phi_0 = \phi$. Conversely, suppose that positive continuous function $\phi_0(x)$ satisfies the inequality $A\phi_0(x) \leq \phi_0(x)$. By (4.3) we have that for any positive $\phi(x)$ and $x \in [0, L]$, $h(x) \leq A\phi(x)$. Consequently, the monotonicity of A implies that A maps the order interval

$$I = \{\phi(x) \in C[0, L] \,|\, h(x) \leq \phi(x) \leq \phi_0(x),\, 0 \leq x \leq L\}$$

into itself. By Theorems 4.1 and 4.2 the operator $A : I \to I$ is a continuous monotone compact operator. It follows (see [4, p. 581]) that the sequence of iterates $\phi_0, A\phi_0, A^2\phi_0, \ldots$ converges to a fixed point ϕ of A, which as we have seen in the introduction of this section, must satisfy $u(x, T; \phi) = g(x)$, $0 \leq x \leq L$. □

A sufficient condition for the existence of a solution is given by the following result.

Corollary 4.4. *If parameter conditions 2-6 and data properties 1-3 are satisfied and, for some positive constant ϕ_0, the condition*

$$\frac{g''(x) - c(x)g'(x)}{d\int_0^T e^{-\gamma(T-\tau)}\mu'(\tau)\,d\tau} \leq \phi_0 k(x; \phi_0)$$

is satisfied, then the inverse problem (IP) has a solution.

Proof. By Theorem 3.2 $\mu'(t)k(x; \phi_0) \leq u_t(x, t; \phi_0)$, from which it follows that

$$A\phi_0 = \frac{g''(x) - c(x)g'(x)}{d\int_0^T e^{-\gamma(T-\tau)}u_t(x, \tau; \phi_0)\,d\tau} \leq \frac{g''(x) - c(x)g'(x)}{d\int_0^T e^{-\gamma(T-\tau)}\mu'(\tau)\,d\tau\, k(x; \phi_0)} \leq \phi_0.$$

Hence Theorem 4.3 applies. □

The following Corollary is useful for computational purposes.

Corollary 4.5. *If the inverse problem (IP) has a solution, then the sequence of iterates h, Ah, A^2h, … converges to a solution of (IP).*

Proof. If ϕ is a solution, then ϕ is a fixed point of A and, as in Theorem 4.3, we have that $h(x) \leq \phi(x)$. Consequently, the iterates $A^n h$ are all bounded by ϕ and so by the compactness and monotonicity of A, the iterates converge to a solution of (IP). □

It is conceivable that the iterates of the last corollary might converge to a solution *other* than the solution ϕ. However, we show that this is not the case, since solutions to (IP) are unique.

Theorem 4.6. *If parameter conditons 2-6 and data conditions 1-3 are satisfied, then the inverse problem (IP) has at most one solution.*

Proof. Suppose that ϕ_1 and ϕ_2 are two distinct solutions to (IP), that is, if $u_i(x,t) = u(x,t;\phi_i)$, $i = 1,2$, then $u_1(x,T) = u_2(x,T)$. As we saw earlier, it is also the case that $A\phi_i = \phi_i$, $i = 1,2$. There are two possibilities: either one function, say ϕ_1, is bounded above by ϕ_2 on the interval $[0,L]$, or $\phi_1(x) - \phi_2(x)$ has both positive and negative values. In the second case, form the function

$$\phi_3(x) = \min\{\phi_1(x), \phi_2(x)\}.$$

Then $\phi_3(x) \le \phi_1(x)$ and $\phi_3(x) \le \phi_2(x)$ on the interval $[0,L]$, so by monotonicity of A, $A\phi_3 \le \phi_3$. It follows from the proof of Theorem 4.3 that the iterates $\phi_3, A\phi_3, A^2\phi_3, \dots$ form a decreasing sequence of functions converging to a solution ϕ_4 to (IP) which is bounded above by each of ϕ_1, ϕ_2, ϕ_3. So we obtain the first case for the functions $\phi_4(x)$ and $\phi_2(x)$. It remains to consider the first case.

In this case it follows from Theorem 3.3 that $u_2(x,t) \le u_1(x,t)$ on Q_T. As in the proof of Theorem 3.3, set $U_k = \int_0^t e^{-\gamma(t-\tau)} u_k(x,\tau)\,d\tau$, $k = 1,2$, $w = u_1 - u_2$ and $W = U_1 - U_2$. We obtain that $w(x,t)$ solves the equation

$$-w_{xx} + c(x)w_x + d\phi_1(x)w - d\phi_1(x)\gamma \int_0^t e^{-\gamma(t-\tau)} w(x,\tau)\,d\tau = d(\phi_2(x) - \phi_1(x))(U_2)_t. \tag{4.12}$$

with homogeneous boundary conditions (2.11) and (2.12). It is easily seen that U_2 is also a solution to (3.2) for $\phi = \phi_2$ that satisfies the boundary conditions $U_2(0,t) = \int_0^t e^{-\gamma(t-\tau)}\mu(\tau)\,d\tau \equiv M(t)$ and $U_2(L,t) + \beta(U_2)_x(L,t) = 0$. Observe that $M(0) = 0$ and $U_2(x,0) = 0$. Also,

$$M'(t) = \mu(t) - \gamma \int_0^t e^{-\gamma(t-\tau)}\mu(\tau)\,d\tau = \int_0^t e^{-\gamma(t-\tau)}\mu'(\tau)\,d\tau,$$

since $\mu(0) = 0$. Parameter condition 5 requires $\mu'(t) > 0$ for $t > 0$. It follows that $M'(t) > 0$ for $t > 0$. Theorem 3.2 implies that $(U_2)_t(x,t) \ge M'(t)k(x) > 0$ for $t > 0$. Now evaluate (4.12) at $t = T$. The left hand side is equal to

$$-d\phi_1(x)\gamma \int_0^T e^{-\gamma(T-\tau)} w(x,\tau)\,d\tau \le 0,\ x \in [0,L]$$

since $w(x,T) = 0$, but the right hand side is positive for some $x \in [0,L]$. This contradiction proves uniqueness. □

There remains the matter of the stability of the inverse problem (IP). The following example shows that (IP) is not stable.

Example 4.7. Let $\phi_0(x) = 1$ and define an unbounded sequence of continuous positive functions $\phi_n(x)$ as follows: $\phi_n(x) = 1$ for x outside an interval of width $1/(n+1)^2$ centered at $1/2$. Inside this interval, let $\phi_n(x)$ be a linear spike with peak of altitude $(n+2)$ centered at $1/2$. Make any choice of the remaining parameters that satisfies parameter conditions 2-6. Let $u_n(x,t) = u(x,t;\phi_n)$, $n = 0,1,\dots$. Each function $u_n(x,t)$ satisfies (3.2). Let $w_n(x,t) = u_0(x,t) - u_n x,t)$, $n = 1,2,\dots$. Subtract these equations for u_0 and u_n and rearrange terms to obtain that $w_n(x,t)$ satisfies the equation

$$-(w_n)_{xx} + c(x)(w_n)_x + d\phi_0(x)w_n = F(x,t) + \gamma d\phi_0(x) \int_0^t e^{-\gamma(t-\tau)} w_n(x,\tau)\,d\tau$$

where

$$F(x,t) = d(\phi_0(x) - \phi_n(x)) \left\{ \gamma \int_0^t e^{-\gamma(t-\tau)} u_n(x,\tau)\, d\tau - u_n(x,t) \right\}.$$

The function $w_n(x,t)$ also satisfies homogeneous boundary conditions $w_n(0) = 0$ and $w_n(L) + \beta(w_n)_x(L) = 0$. It follows that w_n satisfies the integral equation

$$w_n(x,t) = \int_0^L K(x,\xi;\phi_0) \left\{ F(x,t) + \gamma d\phi_0(x) \int_0^t e^{-\gamma(t-\tau)} w_n(x,\tau)\, d\tau \right\} \mathrm{d}\xi \tag{4.13}$$

Since parameter conditions 1-6 are satisfied, we may apply Theorem 3.3 to obtain that $u_t(x,t;\phi_n) \leq u_t(x,t;\phi_0)$ for $(x,t) \in Q_T$. Integrate this inequality from 0 to t and we obtain that $u_n(x,t) \leq u_0(x,t)$ for $(x,t) \in Q_T$. Let $K_C = \max_{0 \leq x,\xi \leq L} K(x,\xi;\phi_0)$ and $K_U = \max_{(x,t)\in Q_T} u_0(x,t)$. Note that $|\phi_0(x) - \phi_n(x)| \leq n+1$ and that this difference vanishes outside an interval of width $1/(n+1)^2$. Take absolute values of both sides of (4.13), reverse order of integration in the second term and we obtain that

$$|w_n(x,t)| \leq \frac{dK_C\,(\gamma T K_U + K_U)}{n+1} + \gamma d K_C L \int_0^t |w_n(x,\tau)|\, d\tau.$$

An application of Gronwall's inequality yields that

$$|w_n(x,t)| \leq \frac{dK_C K_U\,(\gamma T + 1)}{n+1} e^{\gamma d K_C L t} \leq \frac{dK_C K_U\,(\gamma T + 1)\, e^{\gamma d K_C L T}}{n+1}.$$

The latter term tends to 0 as $n \to \infty$. Therefore, the functions $u_n(x,T) = u(x,T;\phi_n)$ converge to $u_0(x,t) = u(x,t;\phi_0)$ uniformly on $[0,L]$, yet $\|\phi_n - \phi_0\| = n$ tends to ∞ with n.

5. Numerical Examples

In this section we show that the inverse problem (IP) is amenable to numerical computation under modest restrictions on the parameters, and present a few example calculations. We confine our calculations to the function $u(x,t)$ in the solution pair u, a of (IP), since $a(x,t)$ is easily recovered from $u(x,t)$ by (3.1).

5.1. **A Numerical Scheme for an Integro-Differential BVP..**

We first develop an integro-differential-BVP solver which can handle the general boundary value problem (2.7)–(2.9). We assume that all the hypotheses of Theorem 2.2 are met, that is, β, γ and $q(x)$ are positive, $F(x,t)$ and $\mu(t)$ non-negative and all functions continuous. Let $u(x,t)$ be the solution to be computed, and define

$$U(x,t) = \int_0^t e^{-\gamma(t-\tau)} u(x,\tau)\, d\tau. \tag{5.1}$$

Let $\Delta x = L/M$ and $\Delta t = T/N$ be step sizes in x and t to be used in the discretization of this problem, where M and N are positive integers. We view (2.7) and (5.1) as an evolutionary system in the unknowns u and U. The evolutionary aspect of U is made clearer by the observation that

$$
\begin{aligned}
U(x,t) &= \int_0^{t-\Delta t} e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau + \int_{t-\Delta t}^{t} e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau \\
&= e^{-\gamma\Delta t}U(x,t-\Delta t) + \int_{t-\Delta t}^{t} e^{-\gamma(t-\tau)}u(x,\tau)\,d\tau.
\end{aligned} \tag{5.2}
$$

Now set $t_n = n\Delta t$, $n = 1,2,\dots N$ and $x_k = k\Delta x$, $k = 1,2,\dots M$. Define $p_k = p(k\Delta x)$, $q_k = q(k\Delta x)$, $\mu_n = \mu(n\Delta t)$ and $F_{k,n} = F(k\Delta x, n\Delta t)$. Likewise our approximations the exact solutions are

$$
\begin{aligned}
u_{k,n} &\approx u(x_k,t_n) = u(k\Delta x, n\Delta t) \\
U_{k,n} &\approx U(x_k,t_n) = U(k\Delta x, n\Delta t)
\end{aligned}
$$

We shall obtain second order accurate approximations (in both Δx and Δt) to these variables. Therefore, we use centered first and second differences to discretize (2.7) and a trapezoidal method to discretize (5.2). There results the system

$$
-\left(1+p_k\frac{\Delta x}{2}\right)u_{k-1,n} + \left\{2+(\Delta x)^2 q_k\left(1-\frac{\gamma\Delta t}{2}\right)\right\}u_{k,n} - \left(1-p_k\frac{\Delta x}{2}\right)u_{k+1,n} = G_{k,n}, \tag{5.3}
$$

where

$$
G_{k,n} = (\Delta x)^2\left\{F_{k,n} + \gamma e^{-\gamma\Delta t}q_k\left(U_{k,n-1}+\frac{\Delta t}{2}u_{k,n-1}\right)\right\}. \tag{5.4}
$$

We also obtain

$$
U_{k,n} = e^{-\gamma\Delta t}U_{k,n-1} + \frac{\Delta t}{2}e^{-\gamma\Delta t}\left(u_{k,n-1}+u_{k,n}\right). \tag{5.5}
$$

These equations account for the interior nodes (x_k,t_n), $k = 1,\dots,M$ and $n = 1,2,\dots N$. The terms $u_{0,n}$ and $u_{M+1,n}$ are eliminated by using the boundary conditions. The integro-differential boundary value problem is now solved by single step marching method, which is explicit in time and implicit in space. The BVP algorithm can be described as follows: given values of U,u at the $(n-1)$-th time level, solve the linear system (5.3)–(5.4) for values of u at the n-th level, then use (5.5) to solve explicitly for U at the n-th level. The solvability of this system imposes some stability restrictions, namely that $\Delta t < 2/\gamma$ and $\Delta x < 2/\|p\|$. With these restrictions, it can be shown that the BVP algorithm converges to the solution with truncation errors in even powers of the step sizes, so that it is second order in step sizes. Therefore, the algorithm can be computed at step and half step sizes, followed by a single step of Richardson extrapolation to yield a fourth order method.

5.2. **An Algorithm for Operator Calculations.** If one assumes exact data, discretization of the operator A is very straightforward. First, discretize the data function g and argument function ϕ via $g_k = g(k\Delta x)$, $k = 0,1,\dots M+1$, and $\phi_k = \phi(k\Delta x)$, $k = 1,2,\dots,M$. Next, observe that $v(x,t) = u_t(x,t)$ satisfies the equation (3.4), together with boundary conditions $v(0,t) = \mu'(t)$ and $v(L,t) + \beta v_x(L,t) = 0$, so that the function $V(x,t) =) \int_0^t e^{-\gamma(t-\tau)}u_t(x,\tau)\,d\tau$ can be approximated by the output of the BVP algorithm, applied to $v(x,t)$, resulting in approximate node values $V_{k,N}$. Given an argument $\Phi = (\phi_1, \ldots, \phi_M)$, the discretized operator is given by

$$(A_M\Phi)_k = \frac{(1 + c_k\frac{\Delta x}{2})g_{k-1} - 2g_k + (1 - c_k\frac{\Delta x}{2})g_{k+1}}{dV_{k,N}(\Delta x)^2} \tag{5.6}$$

which is valid for $k = 1, 2, \ldots M$. This requires a value for g_{M+1}, which could be obtained by using the flux condition that $g(x)$ satisfies at the right endpoint and a centered difference discretization of $g(M\Delta x) + \beta g'(M\Delta x) = 0$. It also requires g_0, which is determined by the boundary condition $g(0) = \mu(T)$. We can now use Corollary 4.5 to devise a simple method, which we term the IP algorithm, for computing the solution to the inverse problem: let the initial guess Φ_0 be the discretization of $h(x)$ as defined in (4.3). Then use fixed point iteration of (5.6) until convergence. We illustrate the IP algorithm in the following three examples. In these examples, we consider the direct problem (1.1)-(1.5) in its equivalent form as the boundary value problem (3.2), (1.3) and (1.4).

5.3. **Examples.** We conclude with the following examples, which illustrate that the inverse problem (IP) can be solved numerically without difficulty using the algorithm we have outlined above, provided that the various data and parameter hypotheses that we have specified are met, and that the noise level is not too great. These results were generated using MATLAB. Versions of the programs used can be obtained from the authors (e.g. tshores@math.unl.edu).

Example 5.1. In this example, we let $L = 2$, $T = 2$, $\gamma = \beta = 1$, $c(x) = 1 + \cos(3x)/3$, $\phi(x) = 2 - x/2 + x^2\sin(3x)/3$ and $\mu(t) = (1 - e^{-\alpha t})/\alpha$, where $\alpha > 0$. It is easily verified that parameter conditions 2-5 are satisfied. Parameter condition 6 presents some difficulty. One can check that the dividing point α_0 is about 0.2031. For values less than α_0, condition 6 is satisfied, and for larger values, it fails. We test two values, namely $\alpha = 0.2$ and $\alpha = 3.0$. In these experiments, we set $\Delta x = \Delta t = 0.01$. We first used the extrapolated BVP algorithm to obtain the data function $g(x)$ to a higher order of accuracy than the expected accuracy of the solution to the inverse problem. We then used the IP algorithm in tandem with the unextrapolated BVP to compute the solution to this inverse problem. Our initial guess $\phi_0(x)$ in all cases is the function $h(x)$ of (4.3), which is iterate 1 in the figures.

As expected, fixed point iteration with the choice $\alpha = 0.2$ exhibits an approximately linear rate of convergence until about the fifth iteration, where the discretization error causes fixed point iteration to stall. Here the initial guess for $\phi(x)$ is the function $\phi_0(x) = h(x)$ of (4.3). At the fourth iteration, the norm of the error is $\|g - \phi_4\| \doteq 0.0021$. The first four iterates and exact solution $g(x)$ are plotted in Figure 5.3, where the monotone behavior of the operator A is evident. When we halved both Δx and Δt, the final error in the same number of iterations was decreased by a factor of about four, confirming that the BVP algorithm is second order accurate.

Fixed point iteration with the choice $\alpha = 3.0$ exhibits some interesting behavior. Again, the initial guess for $\phi(x)$ is the function $\phi_0(x) = h(x)$ of (4.3). The first four iterates and exact solution $g(x)$ are plotted in Figure 5.1. One sees from these plots that the operator A is no longer monotone, which might be expected since parameter condition 6,which is needed in the comparison theorem 3.3, is violated. In spite of this, convergence of the iterates is rapid. We

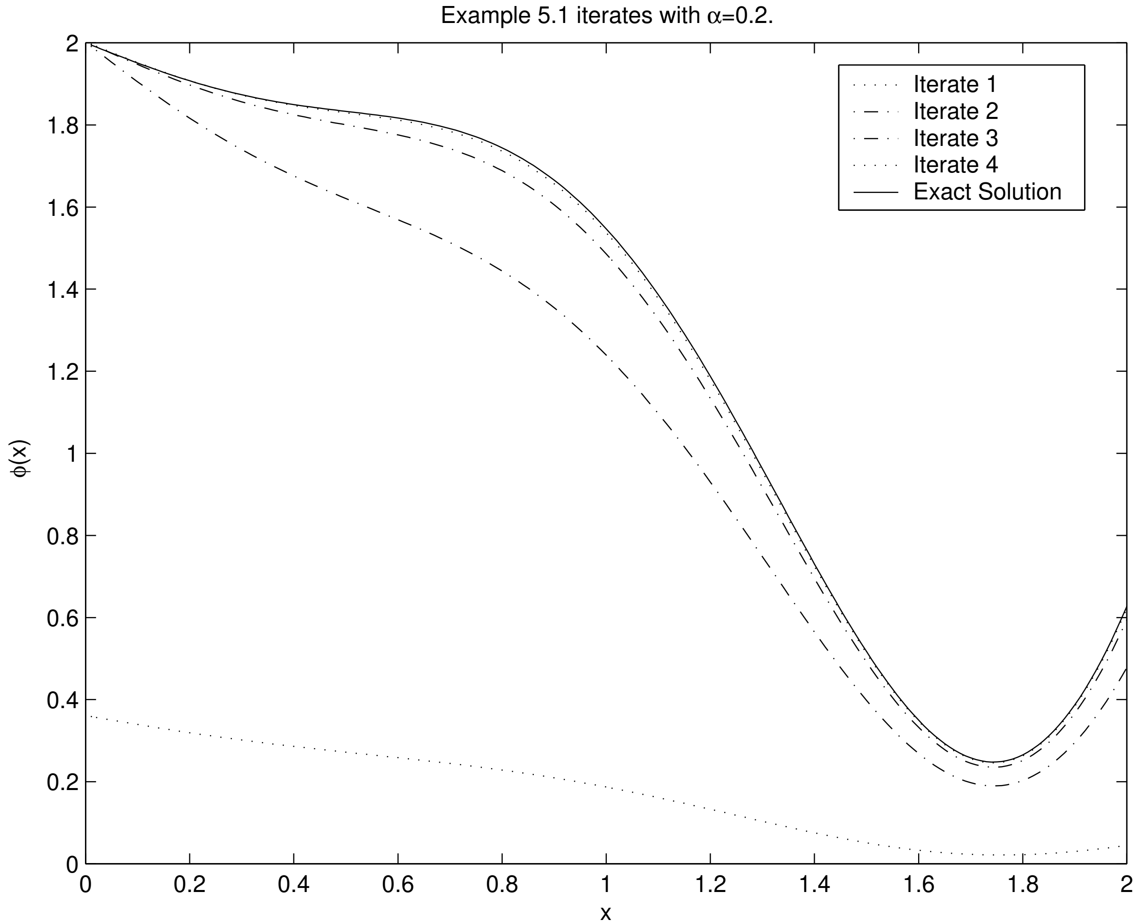

FIGURE 5.1

note, however, that this rapid convergence is to a solution which is visibly less accurate at the left endpoint. We found that halving only Δx did not improve this behavior, while halving only Δt reduced the error by a factor of about two, which was the same result obtained from halving both Δt and Δx. These experiments suggest that the BVP algorithm is possibly second order accurate in space but only first order accurate in time when applied to this problem.

Our next example illustrates the effects of noise on these computations. These effects are quite pronounced. Since the operator A is compact, one needs a regularization strategy. Indeed, the discretization of the definition (4.1) of A by way of (5.6) is itself a regularization strategy with regularization parameter Δx. Since the operator A is nonlinear, an exact analysis of the error is difficult; however, one expects that the principal source of difficulty is in the discretization of the numerator in (4.1). A simple Taylor series analysis shows that if the noise level is bounded by δ, then the discretization error is bounded by a function of the form

$$\frac{\delta}{(\Delta x)^2} + \frac{c_1 \delta}{\Delta x} + c_2 (\Delta x)^2$$

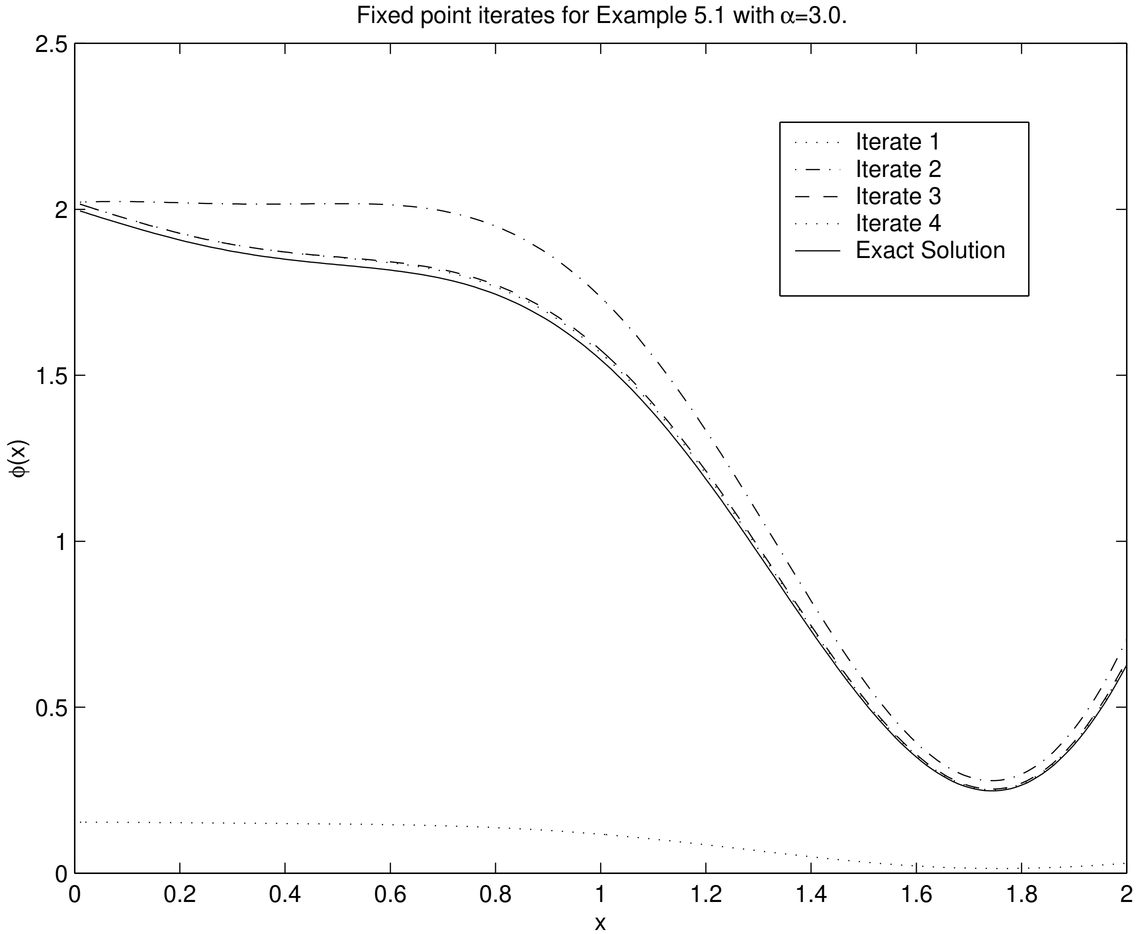

FIGURE 5.2

where c_1 is an upper bound on $|c(x)|$, $0 \leq x \leq L$, and c_2 is a constant that depends on the second and third derivatives of u_t and $c(x)$. This expression is minimized (to leading order in Δx) at $\Delta x = (c_2\delta)^{1/4}$, which supplies us with a regularization strategy. Unfortunately, it is rather pessimistic. There are many alternatives. One could, for example, use Tikhonov regularization on the operator equation $\varphi = A\phi$. Of course, this would alter the numerical form of our calculations significantly. Alternately, one could use some sort of smoothing or filtration of the data in tandem with the discretization we employ in (5.6). A particularly simple minded approach would be to do a least squares fit of the data points to a polynomial of some degree. We illustrate these strategies in the following example.

Example 5.2. In this example we use the same parameters as in Example 5.1, but we restrict α to the value $\alpha = 0.2$, so that all the parameter and data conditions are met. We calculate the data "exactly" using a higher accuracy calculation, and then add noise. The noise is random and uniformly distributed on the interval $[-\delta/2, \delta/2]$. We took $\delta = 0.0001$. In view of the discussion

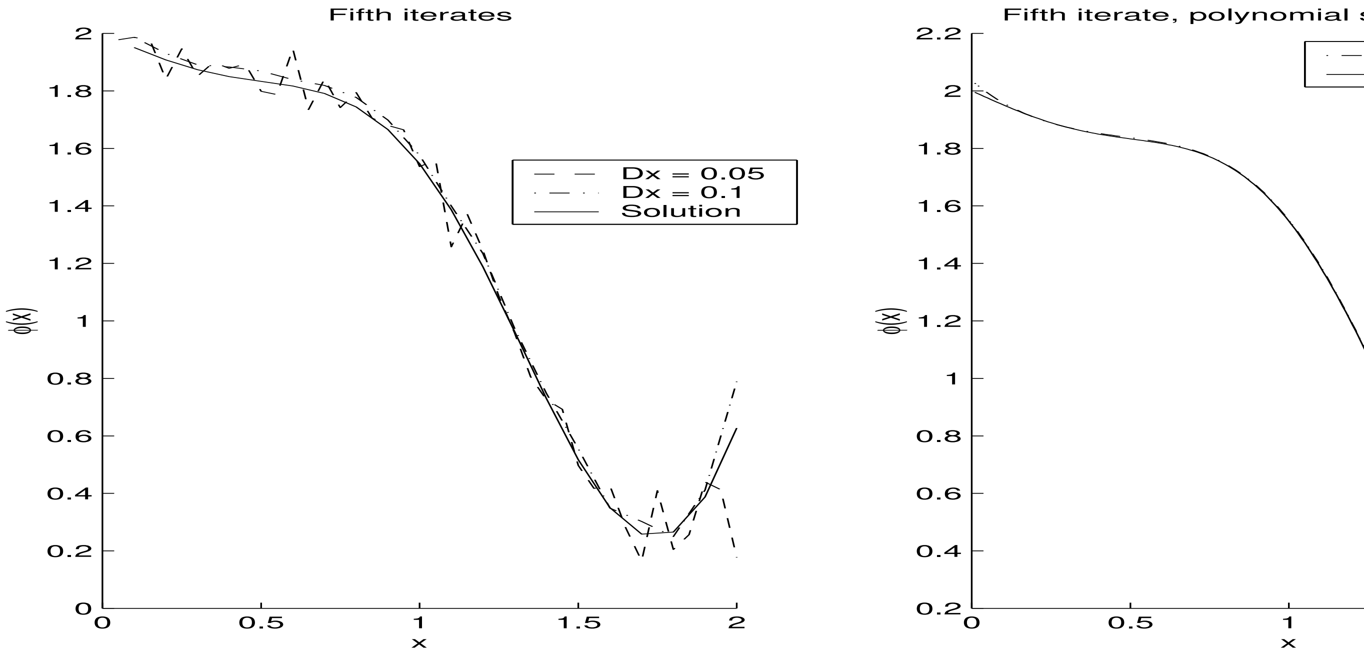

FIGURE 5.3

preceding this example, one might expect the optimum regularization parameter to be $\Delta x \approx 0.1$. Since we are lacking *a priori* derivative information, trial and error will suggest an optimal value. If we use $\Delta x = 0.01$, as in Example 5.1, the results are very poor. Computations for $\Delta x = 0.05$ and $\Delta x = 0.1$ are illustrated in Figure 5.3, along with the exact solution. For purposes of comparison, we then smoothed the data by doing a least squares fit of a tenth degree polynomial to it, then applied our algorithm with $\Delta x = 0.01$. The result is also plotted in Figure 5.3, which clearly indicates that polynomial smoothing is a superior regularization for (IP) in this case.

Example 5.3. For our last example, we let $L = 1$, $T = 3$, $\gamma = \beta = 1$, $c(x) = 1 + \cos(3x)/2$ and $\mu(t) = t^2$. Parameter conditions 2-6 are easily verified. We shall not specify the solution $\phi(x)$, but we will specify $g(x)$. Of course, there are restrictions on our choice, namely the data conditions 1-3 must be satisfied. This example takes on the flavor of a control theory problem, since we are not simulating a measurement of data, but rather specifying the data and attempting to find a $\phi(x)$ which will "steer" the system to the state $u(x,T) = g(x)$. We make the selection $g(x) = \mu(T)e^{-x}$. One checks that data condition 1-3 are satisfied with this choice of $g(x)$. Again, the initial guess for $\phi(x)$ is the function $\phi_0(x) = h(x)$ of (4.3). The first five iterates are shown in Figure ,5.3 and they appear to converge monotonely to a limit. Moreover, it turns out that if $u(x,T;\phi_8)$ is computed and compared to $g(x)$, the norm of the difference is approximately 0.000139.

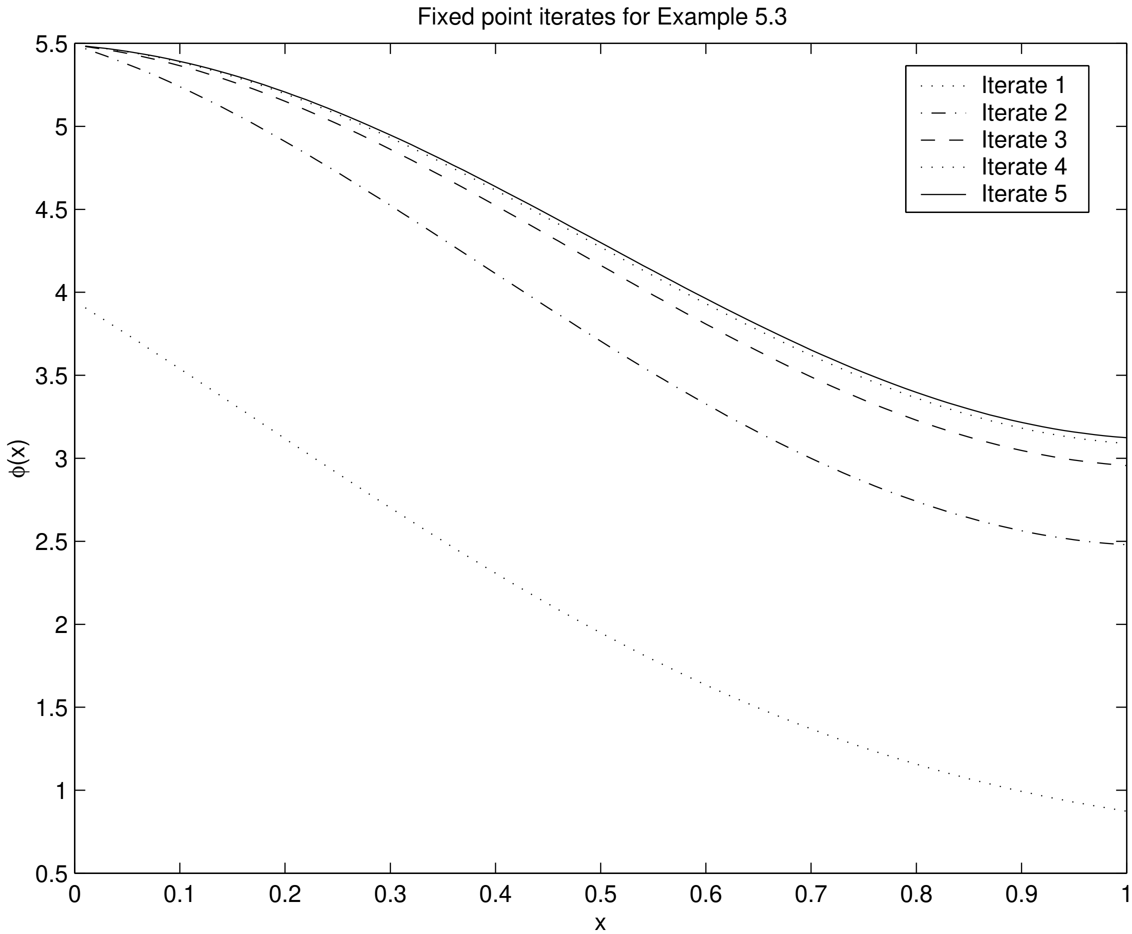

FIGURE 5.4

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