Midterm Exam Math 496, Section 006: Inverse Theory

In-Class Portion

Name: ____________________________ **Score:** ____________

Instructions: Write out your solutions on the paper provided. **Show your work and give reasons for your answers.** NO notes, calculators, laptops, cell phones or other electronic equipment allowed. All problems have equal value. Everyone must do Problem 6. Work any other four problems. *Clearly indicate which four you want graded.* Maximum possible score is 80.

1. Define ill-posed problem and give a specific example of an inverse problem that is ill-posed and reasons for it.

2. Consider the inverse problem

$$\begin{aligned} m_1 - m_2 + m_3 &= 1 \\ m_2 + 2m_3 &= 1 \end{aligned}$$

with coefficient matrix G. Find the normal equations for this problem. What is the rank of this system of normal equations (you don't have to show calculations)?

3. Suppose the inverse problem of solving $G\mathbf{m} = \mathbf{d}$ has right hand side $\mathbf{d}$ whose coordinates have errors which are independent identically distributed random normal variables with mean zero and standard deviation σ. Recall that the r.v. $\chi^2_{obs} = \frac{1}{\sigma^2} \left\| \mathbf{d} - G\mathbf{m} \right\|_2^2 = \sum_{i=1}^{m} \left(d_i - (Gm_{L_2})_i\right)^2 / \sigma^2$ is used to define the p-value $\int_{\chi^2_{obs}}^{\infty} f_{\chi^2}(x)\, dx$. How can we use this statistic?

4. Suppose we want to solve the IFK $d(s) = \int_0^1 g(s, x)\, m(x)\, dx$, given two values $d_i = d(s_i)$, $i = 1, 2$, for the parameter $m(x)$. Set this problem up as a linear system. If you were given sufficient information to solve the system, how would you express the resulting approximation to $m(x)$?

5. What does it mean to say that $G\mathbf{m} = \mathbf{d}$ is a rank deficient linear problem? In such a case, what is the significance of $N(G)$ and $N\left(G^T\right)$?

6. State the SVD Theorem for the $m \times n$ matrix G of rank p . Use it to show that the resolution matrix $G^\dagger G$ is equal to $V_p V_p^T$, where the generalized inverse is given to be $G^\dagger = V_p S_p^{-1} U_p^T$ (explain these terms.)

Take-Home Portion

Points: 50

Due Date: Friday, March 10, 5:00 p.m. for hardcopy, 11:00 p.m. for email.

Instructions: Show your work and give reasons for your answers. There is to be absolutely no consultation of any kind with anyone else other than me about the exam. If there are points of clarification or corrections, I will post them on our message board. ALL materials used in your work that have not been provided by me for this course must be explicitly credited in your write-up. Point values are indicated. You may send an email document (preferably a pdf file, but Word documents will be accepted) or hand in hardcopy at my office.

(6 pts) **1.** Textbook Exercise 1.2

(12 pts) **2.** Textbook Exercise 2.2

(16 pts) **3.** Textbook Exercise 3.2 (Do this exercise with $m = 20$ and $n = 5, 10, 20$. See Note 1 below.)

(16 pts) **4.** Textbook Exercise 4.4. Follow the instructions of this exercise for $m = n = 20$. (See Note 2 below.)

Notes

Note 1: Notice that the my notation differs somewhat from the textbook, page 46. The text assumes that $m = n$ for the representer method. I do not. For this problem, data are given at 20 midpoints, say in the vector s. So, for example, if $n = 5$, then you would use $g_j(x) = g(s_j, x)$, that is, you would only use the first 5 representers, but you would use all the available data, so you represent the parameter function as $m(x) \approx \sum_{j=1}^{5} \alpha_j g_j(x)$ and arrive at a 20×5 system of equations to be solved.

Note 2: Here are a few thoughts about significant digits and accuracy. Suppose that you are given a number in scientific notation, say

$$x_A = d_1.d_2d_3d_4d_5 \ldots \times 10^m$$

and you are told that this number is only accurate to about 3 digits. This means roughly that

$$|x_A - x_T| \approx 0.00e_4e_5 \ldots \times 10^m = e_4.e_5 \ldots \times 10^{m-3}.$$

Thus, we can say that the relative error is

$$\frac{|x_A - x_T|}{|x_T|} = \frac{e_4.e_5 \ldots \times 10^{m-3}}{d_1.d_2d_3d_4d_5 \ldots \times 10^m} \approx 10^{-3},$$

which gives us a nice rule of thumb: if a datum has only n significant digits, then the difference between this number and the true value it is supposed to represent is a relative error of about 10^{-n} and, conversely, if the relative error in an approximation is about 10^{-n} then the approximating number has about n significant digits. Now look again at Equation (4.91) on page 66 of the text and you can see the connection between singular values and significant digits.

Note 3: A handy integration formula for Exercise 3.2:

$$\int_0^1 x^2 e^{-ax} dx = \frac{2e^{-a}}{a^3} \left\{ e^a - 1 - a - \frac{a^2}{2} \right\}$$