Math 4/896: Seminar in Mathematics
Topic: Inverse Theory

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AvH 10

Outline

Basic Theory of SVD

Theorem

(Singular Value Decomposition) Let G be an $m \times n$ real matrix. Then there exist $m \times m$ orthogonal matrix U, $n \times n$ orthogonal matrix V and $m \times n$ diagonal matrix S with diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_q$, with $q = \min\{m, n\}$, such that $U^T GV = S$. Moreover, numbers $\sigma_1, \sigma_2, \ldots, \sigma_q$ are uniquely determined by G.

Definition

With notation as in the SVD Theorem, and U_p, V_p the matrices consisting of the first p columns of U, V, respectively, and S_p the first p rows and columns of S, where σ_p is the last nonzero singular value, then the **Moore-Penrose pseudoinverse** of G is

$$G^\dagger = V_p S_p^{-1} U_p^T \equiv \sum_{j=1}^{p} \frac{1}{\sigma_j} \mathbf{V}_j \mathbf{U}_j^T.$$

Matlab Knows It

Carry out these calculations in Matlab:

```
> n = 6
> G = hilb(n);
> svd(G)
>[U,S,V] = svd(G);
>U'*G*V - S
>[U,S,V] = svd(G,'econ');
> % try again with n=16 and then G=G(1:8)
> % what are the nonzero singular values of G?
```

Applications of the SVD

Use notation above and recall that the null space and column space (range) of matrix G are $N(G) = \{\mathbf{x} \in \mathbb{R}^n \mid G\mathbf{x} = \mathbf{0}\}$ and

$$R(G) = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = G\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n\} = \operatorname{span}\{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n\}$$

Theorem

(1) $\operatorname{rank}(G) = p$ *and* $G = \sum_{j=1}^{p} \sigma_j \mathbf{U}_j \mathbf{V}_j^T$

(2) $N(G) = \operatorname{span}\{\mathbf{V}_{p+1}, \mathbf{V}_{p+2}, \ldots, \mathbf{V}_n\}$, $R\left(G^T\right) = \operatorname{span}\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_p\}$

(3) $N\left(G^T\right) = \operatorname{span}\{\mathbf{U}_{p+1}, \mathbf{U}_{p+2}, \ldots, \mathbf{U}_m\}$, $R(G) = \operatorname{span}\{\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_p\}$

(4) $\mathbf{m}_\dagger = G^\dagger \mathbf{d}$ *is the least squares solution to* $G\mathbf{m} = \mathbf{d}$ *of minimum 2-norm.*

Use the previous notation, so that G is $m \times n$ with rank p and SVD, etc as above. By **data space** we mean the vector space $\mathbb{R}^m$ and by **model space** we mean $\mathbb{R}^n$.

No Rank Deficiency:

This means that $p = m = n$. Comments:

- This means that null space of both G and G^T are trivial (both $\{0\}$).
- Then there is a perfect correspondence between vectors in data space and model space:

$$G\mathbf{m} = \mathbf{d}, \ \mathbf{m} = G^{-1}\mathbf{d} = G^{\dagger}\mathbf{d}.$$

- This is the ideal. But are we out of the woods?
- No, we still have to deal with data error and ill-conditioning of the coefficient matrix (remember Hilbert?).

Heart of the Difficulties with Least Squares Solutions

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Use the notation $\mathbf{m}_\dagger = G^\dagger \mathbf{d}$.

Row Rank Deficiency:

This means that $d = n < m$. Comments:

- This means that null space of G is trivial, but that of G^T is not.
- Here $\mathbf{m}_\dagger$ is the unique least squares solution.
- And $\mathbf{m}_\dagger$ is the actual solution to $G\mathbf{m} = \mathbf{d}$ if and only if $\mathbf{d}$ is in the range of G.
- But *any* least squares solution $\mathbf{m}$ is insensitive to any translation $\mathbf{d} + \mathbf{d}_0$ with $\mathbf{d}_0 \in N\left(G^T\right)$

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Row and Column Rank Deficiency:

This means $p < \min\{m, n\}$. Comments:

- This means that null space of both G and G^T are nontrivial.
- Here $\mathbf{m}_\dagger$ is the least squares solution of minimum length.
- We have trouble in both directions.

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