Math 4/896: Seminar in Mathematics
Topic: Inverse Theory

Instructor: Thomas Shores
Department of Mathematics

AvH 10

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. A. Tarantola, *Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM, Philadelphia, 2004. (Very substantial introduction to inverse theory at the graduate level that emphasises statistical concepts.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems,* Elsivier, New York, 2005. (And the winner is...)

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Outline

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A Motivating Example: Integral Equations

Contanimant Transport

Let $C(x,t)$ be the concentration of a pollutant at point x in a linear stream, time t, where $0 \leq x < \infty$ and $0 \leq t \leq T$. The defining model

$$\begin{aligned}
\frac{\partial C}{\partial t} &= D\frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} \\
C(0,t) &= C_{in}(t) \\
C(x,t) &\rightarrow 0, \quad x \rightarrow \infty \\
C(x,0) &= C_0(x)
\end{aligned}$$

Solution

Solution:

$$C(x,T) = \int_0^T C_{in}(t)\, f(x, T-t)\, dt,$$

where

$$f(x,\tau) = \frac{x}{2\sqrt{\pi D \tau^3}} e^{-(x-v\tau)^2/(4D\tau)}$$

The Inverse Problem

Problem:

Given simultaneous measurements at time T, to estimate the contaminant inflow history. That is, given data

$$d_i = C\left(x_i, T\right),\ i = 1, 2, \ldots, m,$$

to estimate

$$C_{in}\left(t\right),\ 0 \leq t \leq T.$$

More generally

Problem:

Given the IFK

$$d\left(s\right)=\int_{a}^{b}g\left(x,s\right)m\left(x\right)dx$$

and a finite sample of values $d\left(s_i\right)$, $i=1,2,\ldots,m$, to estimate parameter $m\left(x\right)$.

Methods we discuss at the board:

1. Quadrature
2. Representers
3. Other Choices of Trial Functions

Some Methods

More generally

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Outline

1. Chapter 3: Discretizing Continuous Inverse Problems
   - Motivating Example
   - Quadrature Methods
   - Representer Method
   - Generalizations
   - Method of Backus and Gilbert

Quadrature

Basic Ideas:

Approximate the integrals

$$d_i \approx d(s_i) = \int_a^b g(s_i, x)\, m(x)\, dx \equiv \int_a^b g_i(x)\, m(x)\, dx,\ i = 1, 2, \ldots, m$$

(where the representers or data kernels $g_i(x) = g_i(s_i, x)$) by

- Selecting a set of collocation points x_j, $j = 1, 2, \ldots, n$. (It might be wise to ensure $n < m$.)
- Select an integration approximation method based on the collocation points.
- Use the integration approximations to obtain a linear system $G\mathbf{m} = \mathbf{d}$ in terms of the unknowns $m_j \equiv m(x_j)$, $j = 1, 2, \ldots, n$.

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   - Motivating Example
   - Quadrature Methods
   - Representer Method
   - Generalizations
   - Method of Backus and Gilbert

Representers

Rather than focusing on the value of m at individual points, take a global view that $m(x)$ lives in a function space which is spanned by the **representer** functions $g_1(x), g_2(x), \ldots, g_n(x), \ldots$

Basic Ideas:

- Make a selection of the basis functions $g_1(x), g_2(x), \ldots, g_n(x)$ to approximate $m(x)$, say

$$m(x) \approx \sum_{j=1}^{n} \alpha_j g_j(x)$$

- Derive a system $\Gamma \mathbf{m} = \mathbf{d}$ with a Gramian coefficient matrix

$$\Gamma_{i,j} = \langle g_i, g_j \rangle = \int_a^b g_i(x) g_j(x) \, dx$$

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The Most Famous Gramian of Them All:

- Suppose the basis functions turn out to be $g_i(x) = x^{i-1}$, $i = 1, 2, \ldots, m$, on the interval $[0, 1]$.
- Exhibit the infamous Hilbert matrix.

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Other Choices of Trial Functions

Take a still more global view that $m(x)$ lives in a function space which is spanned by a suitable spanning set, which might *not* be the representers!

Basic Ideas:

- Make a selection of the basis functions $h_1(x), h_2(x), \ldots, h_n(x)$ to approximate $m(x)$, say

$$m(x) \approx \sum_{j=1}^{n} \alpha_j h_j(x)$$

- Derive a system $G\alpha = \mathbf{d}$ with a coefficient matrix

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Backus-Gilbert Method

Problem: we want to estimate $m(x)$ at a single point $\widehat{x}$ using the available data, and do it well. How to proceed?

Basic Ideas:

- Write $m(\widehat{x}) \approx \widehat{m} = \sum_{j=1}^{m} c_j d_j$
- Reduce the integral conditions to $\widehat{m} = \int_a^b A(x)\, m(x)\, dx$
- Ideally $A(x) = \delta(x - \widehat{x})$. What's the next best thing? This leads to a quadratic programming problem.

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