Math 4/896: Seminar in Mathematics
Topic: Inverse Theory

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Department of Mathematics

AvH 10

Reminder of Rational File Management

File Management in 384H:

We want to keep perfectly synchronized with all the latest course files. Here's how:

- Create for once and for all a JDEP384hS06 directory somewhere in your own home directory.
- Create subdirectories in which you'll put work, etc., but avoid the names of directories already existing in the course Public directory of the above name.
- Before, or at the start of class, grab a copy of ZipDir.zip which will be located in the current Weekx. Save it in the root of your Math496S06 directory and do NOT unzip it.
- Go into the ZipDir.zip and copy all files inside directory ZipDir.
- Move to root of Math496S06 and paste all. Delete ZipDir.zip.

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Outline

1. Chapter 2: Linear Regression
   - A Motivating Example
   - Solutions to the System
   - Statistical Aspect of Least Squares

The Example

To estimate the mass of a planet of known radius while on the (airless) surface:

Observe a projectile thrown from some point and measure its altitude.

From this we hope to estimate the acceleration a due to gravity and then use Newton's laws of gravitation and motion to obtain from

$$\frac{GMm}{R^2} = ma$$

that

$$M = \frac{aR^2}{G}.$$

A Calculus I problem:

What is the vertical position $y(t)$ of the projectile as a function of time?

Just integrate the constant acceleration twice to obtain

$$y(t) = m_1 + m_2 t - \frac{1}{2} a t^2.$$

We follow the text and write at time t_k, we have observed value d_k and

$$d_k = y(t_k) = m_1 + t\, m_2 - \frac{1}{2} t^2 m_3$$

where $k = 1, 2, \ldots, m$. This is a system of m equations in 3 unknowns. Here

- m_1 is initial y-displacement
- m_2 is initial velocity
- m_3 is acceleration due to gravity.

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Matrix Form

Matrix Form of the System:

$$\text{(Linear Inverse Problem):} \quad G\mathbf{m} = \mathbf{d}.$$

Here $G = \begin{bmatrix} 1 & t_1 & -\frac{t_1^2}{2} \\ 1 & t_2 & \frac{t_2^2}{2} \\ \vdots & \vdots & \vdots \\ 1 & t_m & \frac{t_m^2}{2} \end{bmatrix}$, $\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$, but we shall examine the problem in the more general setting where G is $m \times n$, $\mathbf{m}$ is $n \times 1$ and $\mathbf{d}$ is $m \times 1$.

A Specific Problem

The exact solution:

$$\mathbf{m} = [10, 100, 9.8]^T = (10, 100, 9.8) .$$

Spacial units are meters and time units are seconds. It's easy to simulate an experiment. We will do so assuming an error distribution that is independent and normally distributed with mean $\mu = 0$ and standard deviation $\sigma = 16$.

```
>randn('state',0)
>m = 10
>sigma = 16
>mtrue = [10,100,9.8]'
>G = [ones(m,1), 1:m, -0.5*(1:m).^2]
>datatrue = G*mtrue;
>data = datatrue + sigma*randn(m,1);
```

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Solution Methods

We could try

- The most naive imaginable: we only need three data points. Let's use them to solve for the three variables.
  Let's really try it with Matlab and plot the results for the exact data and the simulated data.
- A better idea: We are almost certain to have error. Hence, the full system will be inconsistent, so we try a calculus idea: minimize the sum of the norm of the residuals. This requires development. The basic problem is to find the **least squares solution** $m = m_{L_2}$ such that

$$\text{(Least Squares Problem):} \quad \|\mathbf{d} - G\mathbf{m}_{L_2}\|_2^2 = \min_{\mathbf{m}} \|\mathbf{d} - G\mathbf{m}\|_2^2$$

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Key Results for Least Squares

Theorem

*The least squares problem has a solution for any $m \times n$ matrix G and data $\mathbf{d}$, namely any solution to the **normal equations***

$$G^T G\mathbf{m} = G^T\mathbf{d}$$

Theorem

If G has full column rank, then the least squares solution is unique, and is given by

$$\mathbf{m}_{L_2} = \left(G^T G\right)^{-1} G^T\mathbf{d}$$

Least Squares Experiments

Let's use Matlab to solve our specific problem with experimental data and plot solutions.
Then let's see why the theorems are true.
There remains:

Problem:

How good is our least squares solution? Can we trust it? Is there a better solution?

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Quality of Least Squares

Theorem

Suppose that the error of ith coordinate of the residual is normally distributed with mean zero and standard deviation σ_i. Let $W = \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_m)$ and $G_W = WG$, $\mathbf{d}_W = W\mathbf{d}$. Then the least squares solution to the scaled inverse problem

$$G_W \mathbf{m} = \mathbf{d}_W$$

*is a **maximum liklihood solution** to the system.*

Theorem

The statistic

$$\|\mathbf{d} - G\mathbf{m}\|_2^2 = \sum^{m} \left(d_i - \left(G m_{L_2}\right)_i\right)^2 / \sigma_i^2$$