# Math 4/896: Seminar in Mathematics
# Topic: Inverse Theory

Instructor: Thomas Shores
Department of Mathematics

AvH 10

Some references:


1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. A. Tarantola, *Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM, Philadelphia, 2004. (Very substantial introduction to inverse theory at the graduate level that emphasises statistical concepts.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems,* Elsivier, New York, 2005. (And the winner is...)

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. A. Tarantola, *Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM, Philadelphia, 2004. (Very substantial introduction to inverse theory at the graduate level that emphasises statistical concepts.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems,* Elsivier, New York, 2005. (And the winner is...)

Some references:


1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. A. Tarantola, *Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM, Philadelphia, 2004. (Very substantial introduction to inverse theory at the graduate level that emphasises statistical concepts.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems*, Elsivier, New York, 2005. (And the winner is...)

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. A. Tarantola, *Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM, Philadelphia, 2004. (Very substantial introduction to inverse theory at the graduate level that emphasises statistical concepts.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems*, Elsivier, New York, 2005. (And the winner is...)

Some references:


1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. A. Tarantola, *Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM, Philadelphia, 2004. (Very substantial introduction to inverse theory at the graduate level that emphasises statistical concepts.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems,* Elsivier, New York, 2005. (And the winner is...)

# Outline

# Preliminaries

1. Let's start with questions on Matlab and the Matlab tutorial, with a brief overview of the part of the tutorial we did not cover.
2. Save a copy of `JDEP384hLecture3.pdf` and `ProbStatLecture-384H.pdf` to your local drives.
3. Save copies of all m files in Week2 to your local drive.
4. Open up the pdf files and fire up Matlab. Get help on addpath and use it to put your m files in Matlab's path.

**Credit and Thanks:** The m files we're using today are found on the outstanding site of James P. LeSage at `http://www.spatial-econometrics.com/`

1. Let's start with questions on Matlab and the Matlab tutorial, with a brief overview of the part of the tutorial we did not cover.
2. Save a copy of `JDEP384hLecture3.pdf` and `ProbStatLecture-384H.pdf` to your local drives.
3. Save copies of all m files in Week2 to your local drive.
4. Open up the pdf files and fire up Matlab. Get help on addpath and use it to put your m files in Matlab's path.

**Credit and Thanks:** The m files we're using today are found on the outstanding site of James P. LeSage at `http://www.spatial-econometrics.com/`

# Preliminaries

1. Let's start with questions on Matlab and the Matlab tutorial, with a brief overview of the part of the tutorial we did not cover.
2. Save a copy of `JDEP384hLecture3.pdf` and `ProbStatLecture-384H.pdf` to your local drives.
3. Save copies of all m files in Week2 to your local drive.
4. Open up the pdf files and fire up Matlab. Get help on addpath and use it to put your m files in Matlab's path.

**Credit and Thanks:** The m files we're using today are found on the outstanding site of James P. LeSage at `http://www.spatial-econometrics.com/`

# Preliminaries

1. Let's start with questions on Matlab and the Matlab tutorial, with a brief overview of the part of the tutorial we did not cover.
2. Save a copy of `JDEP384hLecture3.pdf` and `ProbStatLecture-384H.pdf` to your local drives.
3. Save copies of all m files in Week2 to your local drive.
4. Open up the pdf files and fire up Matlab. Get help on `addpath` and use it to put your m files in Matlab's path.

**Credit and Thanks:** The m files we're using today are found on the outstanding site of James P. LeSage at
`http://www.spatial-econometrics.com/`

# Outline

# Discrete Example

After viewing the probability discussion in ProbStatLecture:

**Dart board experiment:**

The dart board consists of six regions of equal area, and dart is thrown without bias to any region.

Answer these questions:

1. What is the probability of landing in any one region?
2. Suppose the experiment is repeated once. What is the probability of the event of both darts landing in the same region?

# Simulating an Experiment

Let's simulate the dart experiment and graph the results of our experiments using Matlab. Type in

```
> N=36
> x = rand(N,1)*6;
> hist(x,0.5:5.5)
```

Now repeat with larger N.

# Outline

# Uniform Distribution

After viewing the statistics discussion through expectation and variance in ProbStatLecture:

**Uniform distribution:**

Let's take the case of $[a, b] = [0, 1]$.

Answer these questions:

1. What does the graph of the p.d.f. look like?
2. How does simple calculus help us find the c.d.f.?
3. Can we calulate the expectation and variance of the distribution?
4. Can we verify a simple property of expectation and variance from definition?

# Normality and Central Limit Theorem

## Normal distributions:

Let's focus on the standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ -\infty < x < \infty$$

After viewing discussion of normal distributions and the Central Limit Theorem, use addpath to point to the distribution files. Then

```
> x = -10:.1:10;
> help norm_cdf
> y = norm_cdf(x,0,1);
> plot(x,y)
> hold on
> help norm_pdf
> y = norm_pdf(x,0,1);
> plot(x,y)
```

# Outline

# Common Distributions

After viewing the discussion of common distributions in ProbStatLecture:

1. Do a simple plot of the normal distributions $N(0, \sigma)$, $\sigma = 0.5,\ 1.0,\ 2.0,\ 3.0$.
2. Confirm the approximation assertion about Poisson vs binomial by calculating certain values or plotting.
3. Get an idea of the shapes of non-normal distributions as one of their parameters vary.
4. Confirm graphically the limiting assertion about the Student's t distribution.

# Outline

1. Statistics and Probability
   - Probability
   - Statistics
   - Experiments
   - Joint Distributions
   - Parameter Estimation

# Some Conceptual Calculations

After examining the subsection on on joint distributions: Dart throws are independent of each other. $X$ and $Y$ are the location of the dart on $[0, 1]$.

Think about the following:

- What is the joint p.d.f. for these r.v.'s?
- What is the liklihood of achieving a "score" at most 1?
- What is the expected value of the score?

# A Bivariate Normal Distribution

After examining the ProbStatLecture subsection on multivariate normal distributions:

**Asset evaluation (text, p.32):**

Two assets have rates of return $R_1$ and $R_2$that are random variables with means 0.2 and 0.1, variances 0.2 and 0.4, respectively, and covariance $-0.1$. A weighted portfolio had rate of return $R = w_1 R_1 + w_2 R_2$.

Think about the following:

- What is the expectation and variance of $R$?
- If $R_1$ and $R_2$ are jointly bivariate, what does the joint p.d.f. look like?

# Outline

1. Statistics and Probability
   - Probability
   - Statistics
   - Experiments
   - Joint Distributions
   - Parameter Estimation

# Generating Data

After examining the ProbStatLecture section on parameter estimation:
Let's generate data simulating the weighing of an object of weight 10. Assume the experiment is performed 16 times in independent trials. We'll assume the error has a $N(10, .0.01)$ distribution.

```
sigma = sqrt(0.01)
mu = 10
randn('state',0)
data = sigma*randn(16,1)+mu;
```

We'll use this for the next three experiments.

# Estimation of Mean, Known Variance

**Notation:** Given a r.v. $X$ and probability $\alpha$, $x_\alpha$ is the number such that $F_X(x_\alpha) = \alpha$.
Since a c.d.f. is always monotone increasing, we expect that there always is such a number $x_\alpha$, provided that $F_X$ is continuous. In fact

$$x_\alpha = F_X^{-1}(\alpha).$$

**Key Theorem:**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. normal r.v.'s with mean $\mu$ and variance $\sigma^2$. Then the statistic

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

has a standard normal distribution.

A calculation based on this fact shows:

**Confidence interval for $\mu$, $\sigma$ known, confidence coefficient $\alpha$:**

$$\overline{x} + z_{(1-\alpha)/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} - z_{(1-\alpha)/2}\frac{\sigma}{\sqrt{n}}.$$

Here we are using the fact that the standard normal distribution is symmetric about the origin, so that $z_{(1+\alpha)/2} = -z_{(1-\alpha)/2}$. Now use this fact on our data to construct a confidence interval for the true weight of the object. The function `stdn_inv` is helpful here.

# Estimation of Mean, Unknown Variance

**Sampling Theorem:**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. normal r.v.'s with mean $\mu$ and variance $\sigma^2$. Then the statistic

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a Student's t distribution with $n - 1$ degrees of freedom.

A calculation based on this fact shows:

**Confidence interval for $\mu$, $\sigma$ known, confidence coefficient p :**

$$\overline{x} + t_{(1-\alpha)/2}\frac{s}{\sqrt{n}} < \mu < \overline{x} - t_{(1-\alpha)/2}\frac{s}{\sqrt{n}}$$

where $T$ is the Student's t distribution with $n-1$ degrees of freedom.

Here we are using the fact that the t distribution is symmetric about the origin, so that $t_{(1+\alpha)/2} = -t_{(1-\alpha)/2}$. Now use this fact on our data to construct a confidence interval for the true weight of the object. The function `tdis_inv` is helpful here.

# Estimation of Variance

**Key Theorem:**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. normal r.v.'s with mean $\mu$ and variance $\sigma^2$. Then the statistic

$$Y = (n-1)\,\frac{S^2}{\sigma^2}$$

has a chi-square distribution with $n-1$ degrees of freedom.

# Estimation of Mean, Unknown Variance

A calculation based on this fact shows:

**Confidence interval for $\sigma$ , confidence coefficient $\alpha$:**

$$\frac{(n-1)\,s^2}{\chi^2_{(1+\alpha)/2}} < \sigma^2 < \frac{(n-1)\,s^2}{\chi^2_{(1-\alpha)/2}}$$

where $\chi^2$ is the chi-square distribution with $n-1$ degrees of freedom.

The chi-square distribution is NOT symmetric, whence the form above. Now use this fact on our data to construct a confidence interval for the true weight of the object. The function `chis_inv` is helpful here.