Math 4/896: Seminar in Mathematics
Topic: Inverse Theory

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AvH 10

Choice of Regularization Parameter

What should we do about α? This is one of the more fundamental (and intriguing) problems of inverse theory. Let's analyze one of our simple systems for insight, say

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Invariably, our input data for the inverse problem, $(1, 1)$, has error in it, say we have $(1 + \delta_1, 1 + \delta_2)$ for data instead. Let $\delta = \delta_1 + \delta_2$. The regularized system becomes

$$\begin{bmatrix} 2+\alpha & 2 \\ 2 & 2+\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2+\delta \\ 2+\delta \end{bmatrix} = (2+\delta) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which has unique solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2+\alpha & 2 \\ 2 & 2+\alpha \end{bmatrix}^{-1} (2+\delta) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2+\delta}{4+\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Observe that if the input error δ were 0, all we would have to do is let $\alpha \to 0$ and we would get the valid solution $\frac{1}{2}(1,1)$. But given that the input error is not zero, taking the limit as $\alpha \to 0$ gives us a worse approximation to a solution than we would otherwise get by choosing $\alpha \approx 2\delta$. (Our solutions always satisfy $x_1 = x_2$, so to satisfy $x_1 + x_2 = 1$ we need $x_1 = x_2 = \frac{1}{2}$ or as close as we can get to it.) There are many questions here, e.g., how do we know in general what the best choice of regularization parameter is, if any? This and other issues are the subject matter of a course in inverse theory.

Stability

In this special case, we get stability for free – for each *regularized* problem. We cannot hope to have stability for the unregularized problem $A\mathbf{x} = \mathbf{b}$ since A^{-1} doesn't even exist.
But things are even more complicated: For the general linear problem$Kx = y$, even if K^{-1} is well defined the inverse problem may not be stable (although stability happens in some cases). However, we have to look to infinite dimensional examples such as our differentiation example (operator K is integration), where it can be shown that K^{-1} (differentiation) exists but is not continuous, even though K is.

A Continuous Inverse Problem

Let $K : C[0,1] \to C[0,1]$ via the rule $Kf(x) = \int_0^x f(y)\,dy$ This is a one-to-one function. Measure size by the sup norm:

$$\|f\| = \sup_{0 \le x \le 1} |f(x)|$$

so that the “closeness” of $f(x)$ and $g(x)$ is determined by the number $\|f - g\|$. Then one can show that the operator K is continuous in the sense that if $f(x)$ and $g(x)$ are close, then so are $Kf(x)$ and $Kg(x)$.

Let $R = K(C[0,1])$, the range of K. Then $K^{-1} : R \to C[0,1]$ is also one-to-one. But it is not continuous.

Failure of Stability

Consider the function

$$g_\varepsilon(x) = \varepsilon \sin\left(\frac{x}{\varepsilon^2}\right),$$

where $\varepsilon > 0$. We have $\|g_\varepsilon\| = \|g_\varepsilon - 0\| \leq \varepsilon$. So for small ε, $g_\varepsilon(x)$ is close to the zero function. Yet,

$$K^{-1}g_\varepsilon(x) = g_\varepsilon(x)' = \frac{\varepsilon}{\varepsilon^2}\cos\left(\frac{x}{\varepsilon^2}\right) = \frac{1}{\varepsilon}\cos\left(\frac{x}{\varepsilon^2}\right)$$

so that $\left\|K^{-1}g_\varepsilon\right\| = \frac{1}{\varepsilon}$, so that $K^{-1}g_\varepsilon$ becomes far from zero as $\varepsilon \to 0$. Hence K^{-1} is not a continuous operator.

A General Framework

Forward Problem

consists of

- A model fully specified by (physical) parameters m.
- A known function G that, *ideally*, maps parameters to data d by way of

$$d = G(m).$$

(Pure) Inverse Problem

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(Practical) Inverse Problem

is to find m given observations $d = d_{true} + \eta$ so that equation to be inverted is

$$d = G\left(m_{true}\right) + \eta.$$

What we are tempted to do is invert the equation

$$d = G\left(m_{approx}\right)$$

and be happy with m_{approx}. Unfortunately, m_{approx} may be a poor approximation to m_{true}. This makes our job a whole lot tougher – and interesting!

Volterra Integral Equations

Our continuous inverse problem example is a special case of this important class of problems:

Definition

An equation of the form

$$d(s) = \int_a^s g(s, x, m(x))\, dx$$

is called a *Volterra integral equation of the first kind* (VFK). It is *linear* if

$$g(s, x, m(x)) = g(s, x) \cdot m(x)$$

in which case $g(s, x)$ is the *kernel* of the equation. Otherwise it is a *nonlinear* VFK.

In our example $d(s) = \int_a^s m(x)\, dx$, so $g(s, x) = 1,\ a = 0$.

Fredholm Integral Equations of the First Kind (IFK)

Another important class of problems:

Definition

An equation of the form

$$d\left(s\right)=\int_{a}^{b}g\left(s,x,m(x)\right)\,dx$$

is called a *Fredholm integral equation of the first kind* (IFK). It is *linear* if

$$g\left(s,x,m(x)\right)=g\left(s,x\right)\cdot m(x)$$

in which case $g\left(s,x\right)$ is the *kernel* of the equation. If, further,

$$g\left(s,x\right)=g(s-x)$$

the equation is called a *convolution* equation.

Example

Consider our example $d(s) = \int_0^s m(x)\ dx$, again. Define the Heaviside function $H(w)$to be 1 if w is nonnegative and 0 otherwise. Then

$$d(s) = \int_0^s m(x)\ dx = \int_0^\infty H(s-x)\, m(x)\ dx.$$

Thus, this Volterra integral equation can be viewed as a IFK and a convolution equation as well with convolution kernel $g(s,x) = H(s-x)$.

Text Example

Gravitational anomaly at ground level due to buried wire mass

where

- Ground level is the x-axis.
- $h(x)$: the depth of the wire at x.
- $\rho(x)$: is the density of the wire at x.
- $d(s)$: measurement of the anomaly at position s, ground level.

This problem leads to linear and (highly) nonlinear inverse problems

For this portion of the lecture, open copies of the two files LinearAlgebraLecture-496s6.pdf and MatlabLecture-496s6.pdf. We will work our way through the Matlab tutorial, especially the linear algebra parts, with occasional reference to the linear algebra notes when we get to unfamiliar functions or operations.