Math 4/896: Seminar in Mathematics
Topic: Inverse Theory

Instructor: Thomas Shores
Department of Mathematics

AvH 10

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. C. Vogel, *Computational Methods for Inverse Problems*, SIAM, Philadelphia, 2002. (Excellent analysis of computational issues.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems,* Elsivier, New York, 2005. (And the winner is...)

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. C. Vogel, *Computational Methods for Inverse Problems*, SIAM, Philadelphia, 2002. (Excellent analysis of computational issues.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems*, Elsivier, New York, 2005. (And the winner is...)

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. C. Vogel, *Computational Methods for Inverse Problems*, SIAM, Philadelphia, 2002. (Excellent analysis of computational issues.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems*, Elsivier, New York, 2005. (And the winner is...)

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. C. Vogel, *Computational Methods for Inverse Problems*, SIAM, Philadelphia, 2002. (Excellent analysis of computational issues.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems*, Elsivier, New York, 2005. (And the winner is...)

Some references:

1. C. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1993. (A charmer!)
2. M. Hanke and O. Scherzer, *Inverse Problems Light: Numerical Differentiation*, Amer. Math. Monthly, Vol 108 (2001), 512-521. (Entertaining and gets to the heart of the matter quickly)
3. A. Kirsch, *An Introduction to Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996. (Harder! Definitely a graduate level text)
4. C. Vogel, *Computational Methods for Inverse Problems*, SIAM, Philadelphia, 2002. (Excellent analysis of computational issues.)
5. R. Aster, B. Borchers, C. Thurber, *Estimation and Inverse Problems,* Elsivier, New York, 2005. (And the winner is...)

Outline

Course Policy

Course: Math 4/896, Mathematics Seminar
Topic: Inverse Theory
Place/Time: 12 AvH, 3:30-4:45 TR, Spring 2006
Preq: Math 314 (Linear Algebra) and 221 (Differential Equations), or permission.
Instructor: Dr. Thomas Shores
Telephone: Office 472-7233 Home 489-0560
Email: tshores@math.unl.edu
Web Home Page: `http://www.math.unl.edu/~tshores/`
Office Hours: Monday 2:00-4:00, Tuesday 11:00-12:30, Thursday 1:30-3:00, Friday 8:30-10:30, and by appointment. Office: 229 AvH

Objectives

Objectives: To help students achieve competence in the following areas:

- Understanding of mathematical formulations of inverse problems and parameter identification.
- Understanding of inverse theory philosophy and methodology.
- Implementation of algorithms for illustration of inverse problems and their solutions.
- Efficiency and reliability of algorithms.
- Basic skill in using Matlab as a tool for mathematical experimentation.
- Interpretation of results obtained by theoretical analysis and computation (consistent with Hamming's dictum: *The purpose of computing is insight, not numbers.*)

Objectives

Objectives: To help students achieve competence in the following areas:

- Understanding of mathematical formulations of inverse problems and parameter identification.
- Understanding of inverse theory philosophy and methodology.
- Implementation of algorithms for illustration of inverse problems and their solutions.
- Efficiency and reliability of algorithms.
- Basic skill in using Matlab as a tool for mathematical experimentation.
- Interpretation of results obtained by theoretical analysis and computation (consistent with Hamming's dictum: *The purpose of computing is insight, not numbers.*)

Objectives

Objectives: To help students achieve competence in the following areas:

- Understanding of mathematical formulations of inverse problems and parameter identification.
- Understanding of inverse theory philosophy and methodology.
- Implementation of algorithms for illustration of inverse problems and their solutions.
- Efficiency and reliability of algorithms.
- Basic skill in using Matlab as a tool for mathematical experimentation.
- Interpretation of results obtained by theoretical analysis and computation (consistent with Hamming's dictum: *The purpose of computing is insight, not numbers.*)

Objectives

Objectives: To help students achieve competence in the following areas:

- Understanding of mathematical formulations of inverse problems and parameter identification.
- Understanding of inverse theory philosophy and methodology.
- Implementation of algorithms for illustration of inverse problems and their solutions.
- Efficiency and reliability of algorithms.
- Basic skill in using Matlab as a tool for mathematical experimentation.
- Interpretation of results obtained by theoretical analysis and computation (consistent with Hamming's dictum: *The purpose of computing is insight, not numbers.*)

Objectives

Objectives: To help students achieve competence in the following areas:

- Understanding of mathematical formulations of inverse problems and parameter identification.
- Understanding of inverse theory philosophy and methodology.
- Implementation of algorithms for illustration of inverse problems and their solutions.
- Efficiency and reliability of algorithms.
- Basic skill in using Matlab as a tool for mathematical experimentation.
- Interpretation of results obtained by theoretical analysis and computation (consistent with Hamming's dictum: *The purpose of computing is insight, not numbers.*)

Objectives

Objectives: To help students achieve competence in the following areas:

- Understanding of mathematical formulations of inverse problems and parameter identification.
- Understanding of inverse theory philosophy and methodology.
- Implementation of algorithms for illustration of inverse problems and their solutions.
- Efficiency and reliability of algorithms.
- Basic skill in using Matlab as a tool for mathematical experimentation.
- Interpretation of results obtained by theoretical analysis and computation (consistent with Hamming's dictum: *The purpose of computing is insight, not numbers.*)

Student Tasks

Class Attendance: Is required. If absent, it is incumbent upon the student to determine what has been missed as soon as possible. It is advisable to consult with the instructor.

Homework/Projects: Homework will be assigned in class and collected weekly, usually on Mondays, and will be usually returned within one week. Some of the homework assignment problems will be graded in detail for homework points. Although collaboration in solving problems is encouraged as a group effort, it is strictly forbidden to copy someone else's homework. Homework is to be written up individually. The official programming language for this course is Matlab. Prior experience in Matlab is not required. Students will be able to use the Mathematics Computer Lab for computer related exercises. Current information about the course will be available through this lab account and the web (via the 496 homepage or my home page). Using the web is required for keeping track of due dates for homework collections and current activities in

Outline

Universal Examples

What are we talking about? A *direct problem* is the sort of thing we traditionally think about in mathematics:

$$\text{Question} \longrightarrow \text{Answer}$$

An *inverse problem* looks like this:

$$\text{Question} \longleftarrow \text{Answer}$$

Actually, this schematic doesn't quite capture the real flavor of inverse problems. It should look more like

$$\text{Question} \longleftarrow \text{(Approximate) Answer}$$

Universal Examples

Example

(Plato) In the allegory of the cave, unenlightened humans can only see shadows of reality on a dimly lit wall, and from this must reconstruct reality.

Example

The game played on TV show "Jeopardy": given the answer, say the question.

Universal Examples

Example

(Plato) In the allegory of the cave, unenlightened humans can only see shadows of reality on a dimly lit wall, and from this must reconstruct reality.

Example

The game played on TV show "Jeopardy": given the answer, say the question.

Math Examples

Matrix Theory: The $m \times n$ matrix A, $n \times 1$ vector $\mathbf{x}$ and $m \times 1$ vector $\mathbf{b}$ satisfy $A\mathbf{x} = \mathbf{b}$.

- **Direct problem:** given $A, \mathbf{x}$ compute $\mathbf{b}$.
- **Inverse problem:** given $A, \mathbf{b}$, compute $\mathbf{x}$.

Differentiation: given $f(x) \in C[0,1]$ and $F(x) = \int_0^x f(t)\,dt$

- **Direct problem:** given $f(x) \in C[0,1]$, find the indefinite integral $F(x)$.
- **Inverse problem:** given $F(0) = 0$ and $F(x) \in C^1[0,1]$, find $f(x) = F'(x)$.

Math Examples

Matrix Theory: The $m \times n$ matrix A, $n \times 1$ vector $\mathbf{x}$ and $m \times 1$ vector $\mathbf{b}$ satisfy $A\mathbf{x} = \mathbf{b}$.

- **Direct problem:** given $A, \mathbf{x}$ compute $\mathbf{b}$.
- **Inverse problem:** given $A, \mathbf{b}$, compute $\mathbf{x}$.

Differentiation: given $f(x) \in C[0,1]$ and $F(x) = \int_0^x f(t)\, dt$

- **Direct problem:** given $f(x) \in C[0,1]$, find the indefinite integral $F(x)$.
- **Inverse problem:** given $F(0) = 0$ and $F(x) \in C^1[0,1]$, find $f(x) = F'(x)$.

Math Examples

Heat Flow in a Rod

Heat flows in a steady state through an insulated inhomogeneous rod with a known heat source and the temperature held at zero at the endpoints. Under modest restrictions, the temperature function $u(x)$ obeys the law

$$-\left(k(x)u'\right)' = f(x), \;\; 0 < x < 1$$

with boundary conditions $u(0) = 0 = u(1)$, thermal conductivity $k(x)$, $0 \leq x \leq 1$ and $f(x)$ determined by the heat source.
Direct Problem: given parameters $k(x), f(x)$, find $u(x) = u(x;k)$.
Inverse Problem: given $f(x)$ and measurement of $u(x)$, find $k(x)$.

Outline

Well-Posed Problems

A **well-posed problem** is characterized by three properties:

1. The problem has a solution.
2. The solution is unique.
3. The solution is *stable*, that is, it varies continuously with the given parameters of the problem.

A problem that is not well-posed is called **ill-posed**. In numerical analysis we are frequently cautioned to make sure that a problem is well posed before we design solution algorithms. Another problem with unstable problems: even if exact answers are computable, suppose experimental or numerical error occurs: change in solution could be dramatic!

Well-Posed Problems

A **well-posed problem** is characterized by three properties:

1. The problem has a solution.
2. The solution is unique.
3. The solution is *stable*, that is, it varies continuously with the given parameters of the problem.

A problem that is not well-posed is called **ill-posed**. In numerical analysis we are frequently cautioned to make sure that a problem is well posed before we design solution algorithms. Another problem with unstable problems: even if exact answers are computable, suppose experimental or numerical error occurs: change in solution could be dramatic!

Well-Posed Problems

A **well-posed problem** is characterized by three properties:

1. The problem has a solution.
2. The solution is unique.
3. The solution is *stable*, that is, it varies continuously with the given parameters of the problem.

A problem that is not well-posed is called **ill-posed**. In numerical analysis we are frequently cautioned to make sure that a problem is well posed before we design solution algorithms. Another problem with unstable problems: even if exact answers are computable, suppose experimental or numerical error occurs: change in solution could be dramatic!

Illustration: a Discrete Inverse Problem

So what's the fuss? The direct problem of computing F from $F = Kf$ is easy and the solution to the inverse problem is $f = K^{-1}F$, right?

Wrong! All of Hadamard's well-posedness requirements fall by the wayside, even for the "simple" inverse problem of solving for $\mathbf{x}$ with $A\mathbf{x} = \mathbf{b}$ a linear system.

Outline

What Goes Wrong?

1. This linear system $A\mathbf{x} = \mathbf{b}$ has no solution:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. This system has infinitely many solutions:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3. This system has no solution for $\varepsilon \neq 0$ and infinitely many for $\varepsilon = 0$, so solutions do not vary continuously with parameter ε:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}$$

What Goes Wrong?

1. This linear system $A\mathbf{x} = \mathbf{b}$ has no solution:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. This system has infinitely many solutions:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3. This system has no solution for $\varepsilon \neq 0$ and infinitely many for $\varepsilon = 0$, so solutions do not vary continuously with parameter ε:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}$$

What Goes Wrong?

1. This linear system $A\mathbf{x} = \mathbf{b}$ has no solution:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. This system has infinitely many solutions:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3. This system has no solution for $\varepsilon \neq 0$ and infinitely many for $\varepsilon = 0$, so solutions do not vary continuously with parameter ε:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}$$

Some Remedies: Existence

We use an old trick: least squares, which finds the $\mathbf{x}$ that minimizes the size of the residual (squared) $\|\mathbf{b} - A\mathbf{x}\|^2$. This turns out to be equivalent to solving the *normal equations*

$$A^T A\mathbf{x} = A^T\mathbf{b},$$

a system which is guaranteed to have a solution. Further, we can see that if $A\mathbf{x} = \mathbf{b}$ has *any* solution, then every solution to the normal equations is a solution to $A\mathbf{x} = \mathbf{b}$. This trick extends to more abstract linear operators K of equations $Kx = y$ using the concept of "adjoint" operators K^* which play the part of a transpose matrix A^T.

Some Remedies: Uniqueness

We “regularize” the problem. We'll illustrate it by one particular kind of regularization, called *Tikhonov* regularization. One introduces a regularization parameter $\alpha > 0$ in such a way that small α give us a problem that is “close” to the original. In the case of the normal equations, one can show that minimizing the modified residual

$$\|\mathbf{b} - A\mathbf{x}\|^2 + \alpha \|\mathbf{x}\|^2$$

leads to the linear system $\left(A^T A + \alpha I\right)\mathbf{x} = A^T\mathbf{b}$, where I is the identity matrix. One can show the coefficient matrix $A^T A + \alpha I$ is always nonsingular. Therefore, the problem has a unique solution.

Choice of Regularization Parameter

What should we do about α? This is one of the more fundamental (and intriguing) problems of inverse theory. Let's analyze one of our simple systems for insight, say

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Invariably, our input data for the inverse problem, $(1,1)$, has error in it, say we have $(1+\delta_1, 1+\delta_2)$ for data instead. Let $\delta = \delta_1 + \delta_2$. The regularized system becomes

$$\begin{bmatrix} 2+\alpha & 2 \\ 2 & 2+\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2+\delta \\ 2+\delta \end{bmatrix} = (2+\delta) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which has unique solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2+\alpha & 2 \\ 2 & 2+\alpha \end{bmatrix}^{-1} (2+\delta) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2+\delta}{4+\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Choice of Regularization Parameter

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2+\alpha & 2 \\ 2 & 2+\alpha \end{bmatrix}^{-1} (2+\delta) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2+\delta}{4+\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Observe that if the input error δ were 0, all we would have to do is let $\alpha \to 0$ and we would get the valid solution $\frac{1}{2}(1,1)$. But given that the input error is not zero, taking the limit as $\alpha \to 0$ gives us a worse approximation to a solution than we would otherwise get by choosing $\alpha \approx 2\delta$. (Our solutions always satisfy $x_1 = x_2$, so to satisfy $x_1 + x_2 = 1$ we need $x_1 = x_2 = \frac{1}{2}$ or as close as we can get to it.) There are many questions here, e.g., how do we know in general what the best choice of regularization parameter is, if any? This and other issues are the subject matter of a course in inverse theory.

Stability

In this special case, we get stability for free – for each *regularized* problem. We cannot hope to have stability for the unregularized problem $A\mathbf{x} = \mathbf{b}$ since A^{-1} doesn't even exist.

But things are even more complicated: For the general linear problem$Kx = y$, even if K^{-1} is well defined the inverse problem may not be stable (although stability happens in some cases). However, we have to look to infinite dimensional examples such as our differentiation example (operator K is integration), where it can be shown that K^{-1} (differentiation) exists but is not continuous, even though K is.

A Continuous Inverse Problem

Let $K : C[0,1] \to C[0,1]$ via the rule $Kf(x) = \int_0^x f(y)\,dy$ This is a one-to-one function. Measure size by the sup norm:

$$\|f\| = \sup_{0 \le x \le 1} |f(x)|$$

so that the "closeness" of $f(x)$ and $g(x)$ is determined by the number $\|f - g\|$. Then one can show that the operator K is continuous in the sense that if $f(x)$ and $g(x)$ are close, then so are $Kf(x)$ and $Kg(x)$.

Let $R = K(C[0,1])$, the range of K. Then $K^{-1} : R \to C[0,1]$ is also one-to-one. But it is not continuous.

Failure of Stability

Consider the function

$$g_\varepsilon(x) = \varepsilon \sin\left(\frac{x}{\varepsilon^2}\right),$$

where $\varepsilon > 0$. We have $\|g_\varepsilon\| = \|g_\varepsilon - 0\| \leq \varepsilon$. So for small ε, $g_\varepsilon(x)$ is close to the zero function. Yet,

$$K^{-1}g_\varepsilon(x) = g_\varepsilon(x)' = \frac{\varepsilon}{\varepsilon^2}\cos\left(\frac{x}{\varepsilon^2}\right) = \frac{1}{\varepsilon}\cos\left(\frac{x}{\varepsilon^2}\right)$$

so that $\left\|K^{-1}g_\varepsilon\right\| = \frac{1}{\varepsilon}$, so that $K^{-1}g_\varepsilon$ becomes far from zero as $\varepsilon \to 0$. Hence K^{-1} is not a continuous operator.

A General Framework

Forward Problem

consists of

- A model fully specified by (physical) parameters m.
- A known function G that, *ideally*, maps parameters to data d by way of
$$d = G(m).$$

(Pure) Inverse Problem

is to find m given observations d.
We hope (!) that this means to calculate $m = G^{-1}(d)$, but what we really get stuck with is

A General Framework

Forward Problem

consists of

- A model fully specified by (physical) parameters m.
- A known function G that, *ideally*, maps parameters to data d by way of

$$d = G(m).$$

(Pure) Inverse Problem

is to find m given observations d.
We hope (!) that this means to calculate $m = G^{-1}(d)$, but what we really get stuck with is

(Practical) Inverse Problem

is to find m given observations $d = d_{true} + \eta$ so that equation to be inverted is

$$d = G\left(m_{true}\right) + \eta.$$

This makes our job a whole lot tougher – and interesting!

Volterra Integral Equations

Our continuous inverse problem example is a special case of this important class of problems:

Definition

An equation of the form

$$d(s) = \int_a^s g(s, x, m(x))\, dx$$

is called a *Volterra integral equation of the first kind* (VFK). It is *linear* if

$$g(s, x, m(x)) = g(s, x)\, m(x)$$

in which case $g(s, x)$ is the *kernel* of the equation. Otherwise it is a *nonlinear* VFK.

In our example $d(s) = \int_a^s m(x)\, dx$, so $g(s, x) = 1$, $a = 0$.

Fredholm Integral Equations of the First Kind (IFK)

Another important class of problems:

Definition

An equation of the form

$$d\left(s\right)=\int_{a}^{b}g\left(s,x,m(x)\right)\,dx$$

is called a *Fredholm integral equation of the first kind* (IFK). It is *linear* if

$$g\left(s,x,m(x)\right)=g\left(s,x\right)m(x)$$

in which case $g\left(s,x\right)$ is the *kernel* of the equation. If, further,

$$g\left(s,x\right)=g(s-x)$$

the equation is called a *convolution* equation.

Example

Consider our example $d(s) = \int_0^s m(x)\ dx$, again. Define the Heaviside function $H(w)$to be 1 if w is nonnegative and 0 otherwise. Then

$$d(s) = \int_0^s m(x)\ dx = \int_0^\infty H(s-x)\,m(x)\ dx.$$

Thus, this Volterra integral equation can be viewed as a IFK and a convolution equation as well.