Assignment 3 Key for CSCE/Math 441

Thomas Shores
Department of Mathematics
University of Nebraska
Fall 2009

Points: 40
Due: Tuesday, November 17

Exercise 12.2. Calculate the coefficients w_0, w_1, x_0 and x_1 that make the approximation

$$\int_0^1 xf(x)\,dx \approx w_0 f(x_0) + w_1 f(x_1)$$

exact when f is any cubic polynomial. Verify your solution by letting f be a general cubic polynomial. (Note that for this exercise you must find the first three orthogonal polynomials with respect to the weight function $w(x) = x$ on the interval $[0,1]$. Then find the zeros of the quadratic for the nodes x_0, x_1 and use Formula (12.11) in the text to compute the weights w_0, w_1.)

Solution. (15) The inner product for $C[0,1]$ that we use here is the weighted innerproduct

$$\langle f, g\rangle = \int_0^1 xf(x)\,g(x)\,dx.$$

We first find the orthogonal polynomials with respect to this weight by setting

$$\begin{aligned}
p_0(x) &= 1\\
p_1(x) &= x - \frac{\langle p_0, x\rangle}{\langle p_0, p_0\rangle} p_0(x) = x - \frac{\int_0^1 x\cdot 1\cdot x\,dx}{\int_0^1 x\cdot 1\cdot 1 dx} 1\\
&= x - \frac{2}{3}\\
p_2(x) &= x^2 - \frac{\langle p_0, x^2\rangle}{\langle p_0, p_0\rangle} p_0(x) - \frac{\langle p_1, x\rangle}{\langle p_1, p_1\rangle} p_1(x)\\
&= x^2 - \frac{\int_0^1 x\cdot 1\cdot x^2\,dx}{\int_0^1 x\cdot 1\cdot 1 dx} 1 - \frac{\int_0^1 x\left(x-\frac{2}{3}\right)x^2\,dx}{\int_0^1 x\left(x-\frac{2}{3}\right)^2 dx}(x - \frac{2}{3})\\
&= x^2 + \frac{3}{10} - \frac{6}{5}x = x^2 - \frac{6}{5}x + \frac{3}{10}.
\end{aligned}$$

The nodes are just the zeros of $p_2(x)$, which are give by

$$x = \frac{\frac{6}{5} \pm \sqrt{\frac{36}{25} - 4\cdot\frac{3}{10}}}{2} = \frac{6 \pm \sqrt{6}}{10}.$$

Take $x_0 = \left(6 - \sqrt{6}\right)/10$ and $x_1 = \left(6 + \sqrt{6}\right)/10$, so that

$$x_1 - x_0 = \frac{\sqrt{6}}{5}.$$

The weights are then given by

$$w_0 = \int_0^1 x \frac{x - x_1}{x_0 - x_1} dx = \frac{1}{4} - \frac{\sqrt{6}}{36}$$
$$w_1 = \int_0^1 x \frac{x - x_0}{x_1 - x_0} dx = \frac{1}{4} + \frac{\sqrt{6}}{36}$$

Now we can check by exact arithmetic or simply by Matlab that

$$w_0 + w_1 = \frac{1}{2} = \int_0^1 x \cdot 1 \, dx$$
$$w_0 x_0 + w_1 x_1 = \frac{1}{3} = \int_0^1 x \cdot x \, dx$$
$$w_0 x_0^2 + w_1 x_1^2 = \frac{1}{4} = \int_0^1 x \cdot x^2 \, dx$$
$$w_0 x_0^3 + w_1 x_1^3 = \frac{1}{5} = \int_0^1 x \cdot x^3 \, dx$$

Thus the formula is exact on any linear combination of these, which amounts to any cubic polynomial.

Exercise N5. Calculate a Pade approximation for $f(x) = e^x$ of total degree $3 = N = n+m$, with $n = 2$ and $m = 1$. (See the ClassroomNotes441.pdf file for an explanation of the notation.) Next find the best cubic approximation to $f(x)$ on $[-1, 1]$. Find the (approximate) max norm of the error with Matlab. Which is better? (See our ClasroomNotes file for more information on Pade approximations.)

Solution. (15) The form of the approximation is

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + p_2 x^2}{q_0 + q_1 x},$$

where $q_0 = 1$ and all higher terms p_i, $i > 2$ q_j, $j > 1$, are zero. Furthermore, we know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The equations that we need for the remaining 4 terms are given as in the ClassroomNotes file as

$$0 = q_0 a_0 - p_0 = 1 \cdot 1 - p_0 = 1 - p_0$$
$$0 = q_0 a_1 + q_1 a_0 - p_1 = 1 + q_1 - p_1$$
$$0 = q_0 a_2 + q_1 a_1 + q_2 a_0 - p_2 = \frac{1}{2} + q_1 + 0 - p_2$$
$$0 = q_0 a_3 + q_1 a_2 + q_2 a_1 + q_3 a_0 - p_3 = \frac{1}{6} + \frac{1}{2} q_1 + 0 - 0$$

which gives $p_0 = 1$, $p_1 - q_1 = 1$, $p_2 - q_1 = \frac{1}{2}$, and $-\frac{1}{2} q_1 = \frac{1}{6}$. It follows that $q_1 = -1/3$, $p_2 = 1/6$, $p_1 = 2/3$ and $p_0 = 1$. Hence the Pade approximation is

$$r(x) = \frac{1 + \frac{2}{3} x + \frac{1}{6} x^2}{1 - \frac{1}{3} x}.$$

Next, we use minimaxd.m to find the best cubic approximation and compare it to the Pade approximation. Here is the transcript:

```
octave:1> x = -1:0.001:1;
octave:2> expx = exp(x);
octave:3> bestcubic = minimaxd(expx,x,3)
bestcubic =
  0.17953 0.54297 0.99567 0.99458
octave:4> cubicerr = norm(expx-polyval(bestcubic,x),inf)
cubicerr = 0.0055284
octave:5> pade = (1+2*x/3+x.^2/6)./(1-x/3);
octave:6> padeerr = norm(expx-pade,inf)
padeerr = 0.031718
```

(An examination of the graphs shows the problem: The Pade approximation does much better on the interval $[-0.5, 0.5]$, but rapidly loses accuracy at points outside this interval about 0, just like a Taylor approximation. On the other hand, if you were to plot the Taylor cubic polynomial, you see that the Pade approximation does much better than the Taylor polynomial.)

Exercise N6. Determine by Matlab experiment how many function evaluations are required to approximate $\int_{-1}^{1} e^x \sin(\pi x)\,dx$ to an accuracy as good as one obtains with a 7 point Gaussian quadrature method, if one uses a trapezoidal method with equally spaced nodes. (Exact answer: $((e - 1/e)\,/\,(\pi + 1/\pi)$.)

Solution. (10) We recode 'myfcn.m' so that the integrand becomes myfcn:

```
retval = exp(x).*sin(pi*x)
```

Then we use the function file GaussInt.m to calculate the 7 point quadrature approximation to the integral. Experimentation with the function file trap.m eventually yields that an 9182 point trapezoidal method just exceeds the accuracy of the Gaussian quadrature result.

```
octave:1> format long
octave:2> gauss7 = GaussInt(@myfcn,[-1,1],7)
gauss7 = 0.679326212604478
octave:3> truevalue = (exp(1)-1/exp(1))/(pi+1/pi)
truevalue = 0.679326183402095
octave:4> abs(gauss7-truevalue)
ans = 2.920238284520e-08
octave:5> n = 9181; x = linspace(-1,1,n); abs(trap(x,myfcn(x))-truevalue)
ans = 2.92069020080277e-08
octave:6> n = 9182; x = linspace(-1,1,n); abs(trap(x,myfcn(x))-truevalue)
ans = 2.92005395419181e-08
```