Assignment1 Key for CSCE/Math 441

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Fall 2009

Points: 35
Due: Friday, September 11

Exercise 1. Show that the subset $\mathcal{P}$ of $C[a,b]$ consisting of all polynomial functions is a subspace of $C[a,b]$ and that the subset $\mathcal{P}_n$ consisting of all polynomials of degree at most n on the interval $[a,b]$ is a subspace of $\mathcal{P}$.

Solution. (5) Certainly the set $\mathcal{P}$ is nonempty, since it contains the zero polynomial. Let $p(x)$, $q(x)$ be polynomials of degree at most n, the larger of the two degrees. By padding with zero coefficients if necessary, we can assume that they have the forms

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ q(x) &= b_0 + b_1 x + \cdots + b_n x^n, \end{aligned}$$

so that adding them together gives

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + c_n)x^n \in \mathcal{P}$$

and multiplying $p(x)$ by the scalar c gives

$$cp(x) = ca_0 + ca_1 x + \cdots + ca_n x^n \in \mathcal{P}.$$

Hence, by the subspace test, $\mathcal{P}$ is a subpace of $C[a,b]$.

The proof that $\mathcal{P}_n$ is a subspace is essentially the same as the one we have just given with $\mathcal{P}$ in place of $\mathcal{P}_n$.

Exercise 2. Show that $\mathcal{P}_n$ is a finite dimensional vector space of dimension n, but that $\mathcal{P}$ is not a finite dimensional space, that is, does not have a finite vector basis (linearly independent spanning set).

Solution. (4) The space $\mathcal{P}_n$ is spanned by the finite set $1, x, x^2, \ldots, x^n$ and is therefore finite dimensional.

If $\mathcal{P}$ were finite dimensional, every polynomial would be a linear combination of a fixed finite number of polynomials. However, addition and scalar multiplication do not increase the degree of a polynomial, so every polynomial would have degree no more than the largest degree that occurs in this spanning set. Since x^n has degree n for arbitrarily large n, this is impossible, so $\mathcal{P}$ cannot be finite dimensional.

Exercise 3. Show that the uniform norm on $C[a,b]$ satisfies the norm properties.

Solution. (4) Let c be a scalar and $f(x), g(x) \in C[a,b]$.

Certainly $\|f\|_\infty \geq 0$, with equality if and only if the largest value of $|f(x)|$on $[a,b]$ is zero, that is $f(x) \equiv 0$.

Next, we have

$$\|cf(x)\| = \max_{a \leq x \leq b} |cf(x)| = \max_{a \leq x \leq b} |c|\,|f(x)| = |c| \max_{a \leq x \leq b} |f(x)| = |c|\,\|f\|_\infty.$$

Finally,

Next, we have

$$\|f(x)+g(x)\| = \max_{a\le x\le b} |f(x)+g(x)| \le \max_{a\le x\le b} \{|f(x)|+|g(x)|\} \le \max_{a\le x\le b} |f(x)| + \max_{a\le x\le b} |g(x)| = \|f\|_\infty + \|g\|_\infty$$

Thus the three norm laws are satisfied.

Exercise 4. Show that for positive r and $\mathbf{v}_0 \in V$, a normed linear space, the ball $B_r(\mathbf{v}_0)$ is a convex set. Show by example that it need not be strictly convex.

Solution. (4) If $\mathbf{v}, \mathbf{w} \in B_r(\mathbf{v}_0)$, then we have

$$\begin{aligned}\|\lambda\mathbf{v} + (1-\lambda)\mathbf{w} - \mathbf{v}_0\| &= \|\lambda\mathbf{v} - \lambda\mathbf{v}_0 + (1-\lambda)\mathbf{w} - (1-\lambda)\mathbf{v}_0\| \\ &\le \|\lambda\mathbf{v} - \lambda\mathbf{v}_0\| + \|(1-\lambda)\mathbf{w} - (1-\lambda)\mathbf{v}_0\| \\ &\le \lambda\|\mathbf{v} - \mathbf{v}_0\| + (1-\lambda)\|\mathbf{w} - \mathbf{v}_0\| \\ &\le \lambda r + (1-\lambda) r = r.\end{aligned}$$

Hence any convex combination is in this set, so it is convex.

A counterexample would be the unit ball $B_1(0)$ in $\mathbb{R}^2$ with the infinity topology. Note that no point along the boundary segment connecting $(1,-1)$ to $(1,1)$ is in the interior of the unit ball, though these two points are in the ball.

Exercise 5. Confirm that $p_1(x) = x$ and $p_2(x) = 3x^2 - 1$ are orthogonal elements of $C[-1,1]$ with the standard inner product and determine whether the following polynomials belong to span $\{p_1(x), p_2(x)\}$ using Theorem 3.17 of ClassroomNotes.

(a) x^2 (b) $1 + x - 3x^2$ (c) $1 + 3x - 3x^2$

Solution. (6) Calculate these products

$$\begin{aligned}\langle p_1, p_2\rangle &= \int_{-1}^{1} x\left(3x^2-1\right)dx = 0 \\ \langle p_1, p_1\rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \\ \langle p_2, p_2\rangle &= \int_{-1}^{1} \left(3x^2-1\right)\left(3x^2-1\right)dx = \frac{8}{5} \\ \left\langle x^2, p_1\right\rangle &= \int_{-1}^{1} x^2 x dx = 0 \\ \left\langle x^2, p_2\right\rangle &= \int_{-1}^{1} x^2\left(3x^2-1\right) = \frac{8}{15} \\ \left\langle 1+x-3x^2, p_1\right\rangle &= \int_{-1}^{1} \left(1+x-3x^2\right)xdx = \frac{2}{3} \\ \left\langle 1+x-3x^2, p_2\right\rangle &= \int_{-1}^{1} \left(1+x-3x^2\right)\left(3x^2-1\right) = -\frac{8}{5} \\ \left\langle 1+3x-3x^2, p_1\right\rangle &= \int_{-1}^{1} \left(1+3x-3x^2\right)xdx = 2 \\ \left\langle 1+3x-3x^2, p_2\right\rangle &= \int_{-1}^{1} \left(1+3x-3x^2\right)\left(3x^2-1\right) = -\frac{8}{5}\end{aligned}$$

Now calculate the combinations as in Theorem 3.17, and we see that the answers are (a) no, (b) yes, (c) yes.

Exercise 6. Show that differentiation is a linear operator on $V = C^{\infty}(\mathbb{R})$, the space of functions defined on the real line that are infinitely differentiable.

SOLUTION. (4) By definition, if $f(x) \in V$, then so is $f'(x)$, so certainly differentiation maps V into V. Now just check the usual properties of differentiation from elementary calculus:

$$\begin{aligned}(f(x) + g(x))' &= f'(x) + g'(x) \\ (cf(x))' &= cf'(x),\end{aligned}$$

which proves the linear properties for the operator of differentiation.

Exercise 7. Show that the operator $T : C[0,1] \to \mathbb{R}$ given by $T(f) = \int_0^1 f(x)\,dx$ is a linear operator.

SOLUTION. (3) By definition, if $f(x) \in V$, then $\int_0^1 f(x)\,dx$ is a real number, so certainly integration maps V into $\mathbb{R}$. Now just check the usual properties of integration from elementary calculus:

$$\begin{aligned}\int_0^1 (f(x) + g(x))\,dx &= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx \\ \int_0^1 cf(x)\,dx &= c\int_0^1 f(x)\,dx,\end{aligned}$$

which proves the linear properties for the operator of integration.

Exercise 8. Show that every normed linear space V with norm $\|\cdot\|$ is a metric space with metric given by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

SOLUTION. (5) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. If $d(\mathbf{u}, \mathbf{v}) = 0$, then $\|\mathbf{u} - \mathbf{v}\| = 0$, from which it follows by the first norm property that $\mathbf{u} = \mathbf{v}$.

Next, by the second property of norms

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|-1(\mathbf{u} - \mathbf{v})\| = \|\mathbf{v} - \mathbf{u}\| = d(\mathbf{u}, \mathbf{v}).$$

Finally, by the triangle inequality for norms and the definition of d we have

$$d(\mathbf{u}, \mathbf{w}) = \|\mathbf{u} - \mathbf{w}\| = \|\mathbf{u} - \mathbf{w} + \mathbf{w} - \mathbf{v}\| \leq \|\mathbf{u} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{v}\| = d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v}),$$

which is the third metric property. So V is a metric space, as required.