Final Exam **Math 314** **Sample**

Name:______________________________ **Score:**____________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is allowed.

---

(20) **1.** The system of equations

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_3 + x_4 &= 1 \\ 2x_1 + 4x_2 + 2x_4 &= 6 \end{aligned}$$

has coefficient matrix A and right hand side $\mathbf{b}$ such that the row-reduced echelon form of $[A|\mathbf{b}]$ is $\begin{bmatrix} 1 & 0 & 0 & -1 & | & 1 \\ 0 & 1 & 0 & 1 & | & 1 \\ 0 & 0 & 1 & 1 & | & 1 \end{bmatrix}$. Use this information to answer the following:

(a) Find a basis for the null space of A.
(b) Find the form of a general solution of the system $A\mathbf{x} = \mathbf{b}$.
(c) Find a basis for the row space of A.
(d) No matter what the right hand side of $\mathbf{b}$ is, this system has solutions. In terms of rank, why do we know this?

(12) **2.** Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Use (a) to solve the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ for $\mathbf{x}$

(22) **3.** Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}$.

(a) Find the reduced row-echelon form and rank of A.
(b) Find a basis for the column space of A.
(c) Determine which of the following vectors is in the column space of A and, if so, express the vector as a linear combination of the columns of A:
$b_1 = [2, 1, 0]^T$, in $b_2 = [2, -3, 3]^T$.

(14) **4.** Use the Gram-Schmidt process on the basis
$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ of the subspace W of $\mathbb{R}^4$ to produce an orthonormal basis of W.

(12) **5.** Calculate the following determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 8 \\ 1 & 4 & 9 & 9 \end{vmatrix}$$

(12) **6.** The set $\left\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\right\}$ is an orthogonal basis of the inner product space $\mathcal{P}_2$ of polynomials of degree at most 2 with the inner product $\langle p, q \rangle = \int_0^1 p(x)\, qx(x)\, dx$. Assume this and find the coordinates of $p(x) = x^2$ with respect to this basis. What is the angle between $p(x)$ and $q(x) = 1$ in this inner product space?

(12) **7.** Find a least squares solution to the inconsistent system

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

(16) **8.** Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) Find all eigenvalues of A.
(b) Find a basis for the eigenspace corresponding to each eigenvalue of A.
(c) Produce an invertible matrix P and diagonal D such that $P^{-1}AP = D$.

(16) **9.** Let $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$. One of the eigenvalues of A is 0.

(a) Find the eigenvalues of A.
(b) Find a basis for the eigenspace corresponding to each eigenvalue of A.
(c) Produce a unitary matrix U and diagonal matrix D such that $U^*AU = D$.

(15) **10.** Let v be a unit column vector in $\mathbb{R}^n$ and $H = I_n - 2vv^T$.

(a) What is the size of the matrix H?
(b) Give the definition of symmetric matrix and prove H is symmetric.
(c) Give the definition of orthogonal and prove H is orthogonal.

(21) **11.** Circle T for true, F for false or do not answer. Each correct answer is worth 3 points, incorrect answer worth -1 points and no answer worth 0, for a minimum of 0 and maximum of 21 points.

T F (a) If $\mathbf{u}$ and $\mathbf{v}$ are elements of the real inner product space V, then $\langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{v} \rangle \geq \langle \mathbf{u}, \mathbf{v} \rangle^2$.
T F (b) Every real matrix is similar to a diagonal matrix.
T F (c) Every orthogonal set of vectors is linearly independent
T F (d) For $n \times n$ matrices A and B, $(AB)^* = A^*B^*$.
T F (e) If the linear system $Ax = \mathbf{b}$ has a unique solution and A is an $m \times n$ matrix, then $n \leq m$.
T F (f) If $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\dim(V) = 2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set.
T F (g) If the linear system $Ax = \mathbf{b}$ has a unique least squares solution then A has full column rank.

(28) **12.** Fill in the blank or give a short answer in the following:

(a) In terms of rank, a linear system of m equations in n unknowns, with $n > m$ and augmented matrix $\widetilde{A} = [A \,|\, \mathbf{b}]$, has infinitely many solutions if:

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 1+i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & i & 1 \end{bmatrix} =$

(c) A set of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ in the vector space V is defined to be linearly independent if:

(d) If A is symmetric matrix, what can you say about the eigenvalues of A?

(e) Find two 2×2 matrices A and B such that $AB = 0$.

(f) Let $\mathbf{u} = [1, 2, 1]^T$ and $\mathbf{v} = [1, -1, 0]$. Then the projection of $\mathbf{u}$ along $\mathbf{v}$ is $\mathbf{p}$ and $\mathbf{u}$ can be written as $\mathbf{p} + \mathbf{x}$ where $\mathbf{x}$ is orthogonal to $\mathbf{v}$. Find $\mathbf{p}, \mathbf{x}$.

(g) Let $\mathbf{u}$=$(2, -1+i, -1)$ and find $\|\mathbf{u}\|_1 \|\mathbf{u}\|_2$ and $\|\mathbf{u}\|_\infty$.