Exam 3 **Math 314** **Fall 2006**

Name: Key Score:

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is allowed.

(25) **1.** Let $\mathbf{v}_1 = (1, 1, 0))$, $\mathbf{v}_2 = (-1, 1, 1)$, $\mathbf{v}_3 = (1/2, -1/2, 1)$ and $\mathbf{v} = (1, 2, -2)$. Ex. 4.3.3

(a) Find the norm of $\mathbf{v}$.

4/ $\|\underline{v}\| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = \underline{\underline{3}}$

(b) Find the cosine of the angle between the vectors $\mathbf{v}$ and $\mathbf{v}_1$.

5/ $\cos\theta = \dfrac{\underline{v} \cdot \underline{v}_1}{\|\underline{v}\| \cdot \|\underline{v}_1\|} = \dfrac{(1,2,-2)\cdot(1,1,0)}{3 \cdot \sqrt{2}} = \dfrac{3}{3\sqrt{2}} = \underline{\underline{\dfrac{1}{\sqrt{2}}}} = \underline{\underline{\dfrac{\sqrt{2}}{2}}}$

(c) Verify the CBS inequality for the pair of vectors $\mathbf{v}, \mathbf{v}_2$.

CBS says $|\underline{v} \cdot \underline{v}_2| \leq \|\underline{v}\| \cdot \|\underline{v}_2\|$.

5/ So $|\underline{v} \cdot \underline{v}_2| = |(1,2,-2)\cdot(-1,1,1)| = |-1+2-2| = |-1| = 1$.

$\|v\| = 3$

$\|\underline{v}_2\| = \sqrt{3}$ and certainly $\underline{\underline{1 < 3 \cdot \sqrt{3}}}$.

(d) Show $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal set (hence a basis of $\mathbb{R}^3$).

$\underline{v}_1 \cdot \underline{v}_2 = (1,1,0)\cdot(-1,1,1) = -1+1+0 = 0$

$\underline{v}_1 \cdot \underline{v}_3 = (1,1,0)\cdot(\tfrac{1}{2}, -\tfrac{1}{2}, 1) = \tfrac{1}{2} - \tfrac{1}{2} + 0 = 0$

5/ $\underline{v}_2 \cdot \underline{v}_3 = (-1,1,1)\cdot(\tfrac{1}{2}, -\tfrac{1}{2}, 1) = -\tfrac{1}{2} - \tfrac{1}{2} + 1 = 0$

So This is orthogonal set of vectors.

(e) Find the coordinates of $\mathbf{v}$ relative to this basis.

By The coordinates Theorem, These are (c_1, c_2, c_3), where

$c_1 = \dfrac{\underline{v} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} = \underline{\underline{\dfrac{3}{2}}}$

6/ $c_2 = \dfrac{\underline{v} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} = \dfrac{(1,2,-2)\cdot(-1,1,1)}{3} = \underline{\underline{-\tfrac{1}{3}}}$

$c_3 = \dfrac{\underline{v} \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} = \dfrac{(1,2,-2)\cdot(\tfrac{1}{2}, -\tfrac{1}{2}, 1)}{\tfrac{1}{4} + \tfrac{1}{4} + 1} = \underline{\underline{-\tfrac{5}{3}}}$

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(10) 2. Set up and solve the normal equations for the system $A\mathbf{x} = \mathbf{b}$, where$A = [\mathbf{v}_1, \mathbf{v}_2]$, $\mathbf{v}_1 = (1,1,0))$, $\mathbf{v}_2 = (-1,1,1)$ and $\mathbf{b} = (2,1,1)$. Is the least squares solution a genuine solution?

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.

5 $A^T \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$A^T \underline{b} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

3 So $A^TA\,\underline{x} = A^T\underline{b}$ gives $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, ie $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$

Hence $A\underline{x} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\underline{x}$ is <u>not</u> a genuine solution. 2

3. (14) Find an eigensystem for the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Give reasons why the matrix A is or is not diagonalizable. Ex. 5.2.1(d)

3 E vals are $\lambda = 1, 1, 1$.

E vecs for $\lambda = 1$: $A - 1I = \begin{bmatrix} 1-1 & 0 & 0 \\ -2 & 1-1 & 0 \\ 1 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[E_{31}]{E_{23}(2)} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

9 General solution is $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So $\mathcal{E}_1(A) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ which is 2-dimensional.

3 Thus, we cannot find 3 l.i. eigenvectors of A, so that A cannot be diagonalized.

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(24) **3.** Matrix $A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$ has eigenvalues -3, 2, and eigenvectors $\mathbf{v}_1 = (2,-1)$, $\mathbf{v}_2 = (1,2)$. Exm 5.4.1(a)

(a) Use this information to find a diagonalizing matrix P for A and resulting diagonal matrix D.

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Let $P = [\underline{v}_1, \underline{v}_2] = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

Check That $AP = [A\underline{v}_1, A\underline{v}_2] = \begin{bmatrix} -6 & 2 \\ 3 & 4 \end{bmatrix} = [-3v_1, 2v_2]$.

So we know That

$$P^{-1}AP = D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}.$$

(b) Use (a) to find a formula for powers of A in terms of powers of eigenvalues of A.

We have $A = PDP^{-1}$ and $A^k = PD^kP^{-1} = P\begin{bmatrix} (-3)^k & 0 \\ 0 & 2^k \end{bmatrix}P^{-1}$.

Now $P^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, so we calculate

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$$A^k = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-3)^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\frac{1}{5} =$$

$$= \frac{1}{5}\begin{bmatrix} 2(-3)^k & 2^k \\ -(-3)^k & 2^{k+1} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4(-3)^k + 2^k, & -2(-3)^k + 2^{k+1} \\ -2(-3)^k + 2^{k+1}, & +\text{[redacted]}(-3)^k + 2^{k+2} \end{bmatrix}$$

(c) Find unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$ in the directions of $\mathbf{v}_1$, $\mathbf{v}_2$, respectively, and exhibit an orthogonal matrix U that diagonalizes A.

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Let $\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\underline{u}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. 3

Set $U = [\underline{u}_1, \underline{u}_2] = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. 4

Then U is orThogonal $(U^T = U^{-1})$ and

$U^TAU = D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ (not required)

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