Exam 2 Math 314 Fall 2006

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Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is allowed.

(30) **1.** Let $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] = \begin{bmatrix} 1 & 4 & 1 & -1 \\ 2 & 4 & 1 & -1 \\ 4 & 8 & 2 & -2 \end{bmatrix}$ with reduced row echelon form $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for $\mathcal{R}(A)$, the row space of A.

6 Nonzero rows of R suffice:

$\{(1,0,0,0), (0,1,-\frac{1}{4},\frac{1}{4})\}$

(b) Find a basis for $\mathcal{C}(A)$, the column space of A.

6 Columns of A corresponding to pivots in R suffice:

$\{(1,2,4), (4,4,8)\}$

(c) Find a basis for $\mathcal{N}(A)$, the null space of A. $= \{\underline{c} \mid A\underline{c} = \underline{0}\}$. General soln is

6 $\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4}c_3 - \frac{1}{4}c_4 \\ c_3 \\ c_4 \end{bmatrix} = c_3 \begin{bmatrix} 0 \\ 1/4 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ -1/4 \\ 0 \\ 1 \end{bmatrix}$. So a basis is $\begin{bmatrix} 0 \\ 1/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/4 \\ 0 \\ 1 \end{bmatrix}$.

(d) Find all possible linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ that sum to $\mathbf{0}$.

6 From (c), $\underline{0} = A\underline{c} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + c_4\underline{v}_4$

So $\underline{0} = 0 \cdot \underline{v}_1 + (\frac{1}{4}c_3 - \frac{1}{4}c_4)\underline{v}_2 + c_3\underline{v}_3 + c_4\underline{v}_4$

where c_3, c_4 are arbitrary.

(e) Which $\mathbf{v}_j$'s are redundant in the list of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$?

6 From (d), we can take $c_3 = -1$, $c_4 = 1$ & get

$-\frac{1}{2}\underline{v}_2 + (-1)\underline{v}_3 + 1 \cdot \underline{v}_4 = \underline{0}$

So $\underline{v}_2, \underline{v}_3, \underline{v}_4$ are redundant, but not $\underline{v}_1$, since it never has nonzero coefficient in a l.c. with value 0.

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(16) **2.** Use the Subspace Test to decide if W is a subspace of the vector space V, where
(a) $V = \mathbb{R}^3$ and $W = \{(a, b, a-b+1) \mid a, b \in \mathbb{R}\}$

(3.2.1)

If $(0,0,0) = (a, b, a-b+1)$, Then $a=0, b=0$, & $0 = 1$, which is not true. So $(0,0,0) \notin W$ and W fails subspace Test

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(b) $V = C[0,1]$, the continuous functions on $[0,1]$ and $W = \{f(x) \mid f(x) \in C[0,1] \text{ and } f(1) = 0\}$.

The zero fcn satisfies $\underline{0}(0) = 0$, so $0 \in W$.

If $f, g \in W$, Then $f(1) = 0 = g(1)$, so $(f+g)(1) = f(1)+g(1) = 0+0 = 0$

Hence $f+g \in W$.

8 If $f \in W$, c is constant, Then $(cf)(1) = c \cdot f(1) = c \cdot 0 = 0$

so $c \cdot f \in W$.

So W passes The subspace test and is subspace of V.

(3.5b) (10) **3.** Assume that $1+x$, $x+x^2$, $1-x$ is a basis of $\mathcal{P}_2$, the space of polynomials of degree at most two, and find the coordinates of $2+x^2$ relative to this basis.

Simply write $(2+x^2) = c_1(1+x) + c_2(x+x^2) + c_3(1-x)$

and get $2+x^2 = (c_1+c_3)\cdot 1 + (c_1+c_2-c_3)x + c_2 \cdot x^2$.

Hence $c_2 = 1$ and $c_1+c_3 = 2$ and $c_1+c_2-c_3 = 0$.

10 So $c_1+c_3 = 2$, $c_1 - c_3 = -c_2 = -1$.

So $(c_1+c_3)+(c_1-c_3) = 2c_1 = 2-1 = 1$.

$\therefore c_1 = \frac{1}{2}$ and so $c_3 = 2 - c_1 = 3/2$.

Matrix approach: solve $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

So coordinates are $c_1 = \frac{1}{2}$, $c_2 = 1$, $c_3 = 3/2$

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2.6.9b) (8) 4. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. Find the adjoint matrix adj (A) of A.

Minors $M(A) = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ } 3

Cofactor Matrix $A_{cof} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{bmatrix}$. } 3

So $\operatorname{adj}(A) = A_{cof}^T = \begin{bmatrix} -1 & 0 & -3 \\ 0 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ } 2

3.5.4) (10) 5. You are given that $\mathbf{w}_1 = (0,1,0)$, $\mathbf{w}_2 = (1,1,1)$ is a linearly independent set in $V = \mathbb{R}^3$ and $\mathbf{v}_1 = (1,3,1)$, $\mathbf{v}_2 = (2,-1,1)$, $\mathbf{v}_3 = (1,0,1)$ is a basis of V. Steinitz substitution says that $\mathbf{w}_1$, $\mathbf{w}_2$ can be substituted into the basis in place of certain $\mathbf{v}_i$'s. Which substitutions work?

A basis of $\mathbb{R}^3$ needs 3 elements, so we need only add one of $\underline{v}_1, \underline{v}_2, \underline{v}_3$ to get one. So we just Test for linear independence:

$\underline{w}_1, \underline{w}_2, \underline{v}_1$: $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -1(1-1) = 0$. So This does not work

$\underline{w}_1, \underline{w}_2, \underline{v}_2$: $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = -1(1-2) = -1$. So These vectors are a basis

w_1, w_2, v_3 : $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -1(1-1) = 0$. So This does not work.

Only substitution: $\underline{w_1, w_2, v_2}$.

[Many other mtds, like general l.c.'s, are possible]

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(16) **6.** Fill in the blanks or answer True/False (T/F).

2 (a) Every vector space is finite dimensional (T/F) F.

2 (b) Elementary row operations on a matrix do not change the column space (T/F) F.

2 (c) If $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{x} = \mathbf{x}_1$ are both vector solutions to the linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_0$ is in the null space of A. (T/F) T.

4 (d) The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (x + y, x - 2y)$ is linear (T/F) T and one-to-one (T/F) T.

2 (e) The Basis Theorem asserts that every finite dimensional vector space has a basis.

2 (f) The Dimension Theorem asserts that any two bases of a f.d.v.s. have same # of elts.

3.5.9.f) (g) If A is a 5×5 matrix and $\det(A) = 2$, then the first 4 columns of A span a 4 dimensional
2 subspace of $\mathbb{R}^5$ (T/F) T.

(10) **7.** (a) Show from definition of linear dependence that the columns of the matrix $\begin{bmatrix} 3 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$ form a linearly dependent set.

Write $\underline{v}_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and

obtain $0\,\underline{v}_1 + 1 \cdot \underline{v}_2 + 0\,\underline{v}_3 = \underline{0}$, so by dfn.

These vectors are l.d. since at least one coef. is not 0,
but sum is zero vector.

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[Also, combinations like $\underline{v}_1 + 0\,\underline{v}_2 - 3v_3 = \underline{0}$, etc]
[OR: Note There are redundant vectors since $\underline{v}_1 = 3\,\underline{v}_3$.]

(b) (*Honors students only*) Prove that any set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V that contains the zero vector is a linearly dependent set.

If some $\underline{v}_i$ is The zero vector, we write

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$0\,\underline{v}_1 + \ldots + 0\,\underline{v}_{i-1} + 1 \cdot \underline{v}_i + 0\,\underline{v}_{i+1} + \ldots + 0\,\underline{v}_n = \underline{0}$

which is true and proves $v_1, \ldots, v_n$ is l.d.
since a nontrivial l.c. has sum $\underline{0}$.

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