Exam 1 **Math 314** **Fall 2006**

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Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate*. You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is allowed.

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(24) **1.** Consider the linear system given by the following:

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 2x_1 + x_2 - 2x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - 2x_4 &= 4 \end{aligned}$$

(a) (12) Use Gauss-Jordan elimination to find the general solution to this system. Clearly specify the elementary row operations you use.

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & -2 & 1 \\ 2 & 2 & 2 & -2 & 4 \end{bmatrix} \xrightarrow[E_{31}(-2)]{E_{21}(-2)} \begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_2(-1)]{E_{12}(1)} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{rhs} \end{array} \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3, x_4 free

So general soln is
$$\left.\begin{aligned} x_1 &= x_3 + x_4 - 1 \\ x_2 &= -2x_3 + 3 \\ x_3, x_4 &\text{ free.} \end{aligned}\right\} 5$$

−3 if not solved, but eqns listed.

(b) (4) If we write the system as $A\mathbf{x} = \mathbf{b}$, what are the coefficient matrix A and right-hand-side vector $\mathbf{b}$? What are the rank and nullity of A?

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 1 & 0 & -2 \\ 2 & 2 & 2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \quad \text{rank } A = 2, \quad \text{nullity } A = 2$$

(c) (3) Express the reduced row echelon form R of the augmented matrix $\widetilde{A}$ of this system as product of elementary matrices and $\widetilde{A}$.

−2 if wrong side

$$3\left\{ R = E_2(-1)\,E_{12}(1)\,E_{31}(-2)\,E_{21}(-2)\,\widetilde{A} \right.$$

(d) (5) Apply the row operations used in part (a) in the same order as in (a) to a general right hand side vector $\mathbf{b} = (b_1, b_2, b_3)$. What is the resulting vector?

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \xrightarrow[E_{31}(-2)]{E_{21}(-2)} \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_1 \end{bmatrix} \xrightarrow{E_{12}(1)} \begin{bmatrix} b_1 + b_2 - 2b_1 \\ -b_2 + 2b_1 \\ b_3 - 2b_1 \end{bmatrix} = \begin{bmatrix} b_2 - b_1 \\ 2b_1 - b_2 \\ b_3 - 2b_1 \end{bmatrix}$$

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(16) **2.** Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Find the inverse of A and use it to solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (2, -4, 8)$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[E_2(\frac{1}{2})]{E_{31}(1)} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{E_{32}(2)} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{E_3(\frac{1}{2})} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{E_{13}(-1)} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{E_{12}(+2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & +1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ 12

Soln is $x = A^{-1}b = \frac{1}{2}\begin{bmatrix} 1 & +1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -10 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 3 \end{bmatrix}$ 4

(12) **3.** Solve the following systems for the (complex) variable z. Express your answere in standard form ($z = x + iy$) where possible.

(a) $z = e^{i\pi} + 2i$. $= \cos\pi + i\sin\pi + 2i$

4 $= -1 + 0 + 2i$

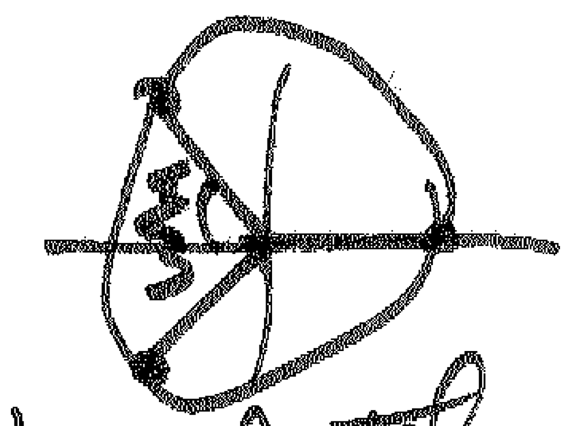
$= -1 + 2i$

(b) $(2 + i)\, z = 1$

4 $z = \frac{1}{(2+i)(2-i)}(2-i) = \frac{1}{4+1}(2-i) = \frac{1}{5}(2-i)$

(c) $z^3 = 1$ $= 1e^{0i + 2k\pi i}$, $j = 0, 1, 2$

So $z = 1 \cdot e^{2k\pi i/3} = 1e^{0i},\ 1 \cdot e^{\frac{2\pi}{3}i},\ 1e^{\frac{4\pi i}{3}}$



$= 1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ -1 if not evaluated

$= 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ 4

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(20) 4. Carry out these calculations or indicate they are impossible. You are given that $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1+i \\ 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) $\mathbf{y}C\mathbf{x}$

4 $[3\ 4]\begin{bmatrix} 2 & 1+i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = [3\ 4]\begin{bmatrix} 4+1+i \\ 1 \end{bmatrix} = 3(5+i)+4$

$= 19+3i$

(b) $\mathbf{xy}$

3 $\begin{bmatrix} 2 \\ 1 \end{bmatrix}[3\ 4] = \begin{bmatrix} 2\cdot3 & 2\cdot4 \\ 1\cdot3 & 1\cdot4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix}$

(c) $\mathbf{x} + 2\mathbf{x}^T$

3 $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2[2, 1]$ Not possible.

(d) D^* $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$.

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(e) C^{-1}

4 $\frac{1}{2\cdot1 - 0\cdot(1+i)}\begin{bmatrix} 1 & -(1+i) \\ -0 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1-i \\ 0 & 2 \end{bmatrix}$.

(f) CD

3 $\begin{bmatrix} 2 & 1+i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1+i & 4 \\ 0 & 1 & 0 \end{bmatrix}$

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Parts (a), (b) are worth 3 pts, rest 2pts.

(18) 5. Fill in the blanks ~~(3 points)~~ or answer True/False ~~(2 point). Each correct answer is worth 2 points, false answer worth -1 points and blank answer worth 0 points for a minimum of 0 and maximum of 16.~~

2 (a) If A is a 2×2 nonzero matrix and the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for some $\mathbf{b}$ then rank $A =$ __1__ and A __is not__ invertible. (Fill in "is" or "is not".)

3 (b) $T((x, y)) = (x + y, 2x, 4y - x)$ is a matrix multiplication function $T_A((x, y))$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 4 \end{bmatrix}$ 1 pt if transp

3 (c) If a Markov chain has transition matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$, and initial state $x^{(0)} = (1, 0)$, then $x^{(2)} =$ $\frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\frac{1}{4}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 2\cdot 2 + 1\cdot 2 \\ 2\cdot 2 + 3\cdot 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

2 (d) As a matrix–vector product, $x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

2 (e) If 3×3 matrix A is invertible, then the reduced row echelon form of A is: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2 (f) Any homogeneous (right-hand-side vector $\mathbf{0}$) linear system is consistent (T/F): T

2 (g) If A, B are 2×2 matrices, then $(AB)^2 = A^2B^2$ (T/F): F

2 (h) Every diagonal matrix is symmetric (T/F): T

(10) 6. Let $A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \end{bmatrix}$.

(a) Verify the commutative law of matrix addition for these two matrices.

10 or 5(H)

$A + B = \begin{bmatrix} -1+1, & 0+2, & -1-1 \\ 0+4, & 1+1, & 2+3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & 5 \end{bmatrix}$

$B + A = \begin{bmatrix} 1-1, & 2+0, & -1-1 \\ 4+0, & 1+1, & 3+2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & 5 \end{bmatrix}$ ← These are equal, as required!

(c) (Honors only) Give a proof of this law.

5(H)

Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $m \times n$ matrices.

1 pt for special case

By dfn, $A + B = [a_{ij} + b_{ij}]$, $m \times n$

and $B + A = [b_{ij} + a_{ij}]$, also $m \times n$

But $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ by cmmt. law of scalar addition.

Hence $A + B = B + A$.

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