**Quiz 4** **Math 221H** **Fall 2008**

**Name:**______________________________ **Score:**__________

*Instructions:* Show your work in the spaces provided below for full credit. You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., $\pi$) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Notes or text in *any* form are not allowed. The only electronic equipment allowed is a calculator.

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(10) **1.** (Exer. 5.2.3) Use the eigenvalue method to find the general solution to the IVP $x_1' = 3x_1 + 4x_2$, $x_2' = 3x_1 + 2x_2$, $x_1(0) = x_2(0) = 1$. (You may *assume* that one eigenvalue is $-1$ with corresponding eigenvector $\mathbf{v} = [1, -1]^T$.)

SOLUTION. The system in question is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with characteristic equation

$0 = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 3\cdot 4 = \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6)$, so $\lambda = -1, 6$.

Eigenvalue $\lambda = 6$ gives eigenvector $\mathbf{v} = (a,b)$ with $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-6 & 4 \\ 3 & 2-6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3a+4b \\ 3a-4b \end{bmatrix}$, so if we allow $b$ to be arbitrary, we obtain $a = \frac{4}{3}b$ $\mathbf{v} = \left(\frac{4}{3}b, b\right) = \left(\frac{4}{3}, 1\right)b$. Take $b$=3 and obtain solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{v}e^{6t} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t}$.

Eigenvalue $\lambda = -1$ gives eigenvector $\mathbf{v} = (a,b)$ with $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3--1 & 4 \\ 3 & 2--1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4a+4b \\ 3a+3b \end{bmatrix}$, so obtain $\mathbf{v} = (b, -b) = (1,-1)b$. Take $b$=1 and obtain solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{v}e^{-t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$.

Thus, the general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t}$. Evaluate at $t = 0$ and use the initial condition to obtain $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 + 4c_2 \\ -c_1 + 3c_2 \end{bmatrix}$. Add equations to obtain $c_2 = \frac{2}{7}$, so $c_1 = \frac{6}{7} - 1 = -\frac{1}{7}$ and the solution is $x_1(t) = -\frac{1}{7}e^{-t} + \frac{8}{7}e^{6t}$, $x_2(t) = \frac{1}{7}e^{-t} + \frac{6}{7}e^{6t}$.

(10) **2.** (Exer. 5.2.8) Use the eigenvalue method to find the general solution to the system $x_1' = x_1 - 5x_2$, $x_2' = x_1 - x_2$.

SOLUTION. The system in question is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with characteristic equation

$0 = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = -(1+\lambda)(1-\lambda) - (-5) = \lambda^2 - 1^2 + 4 = \lambda^2 + 4$ with roots $\lambda = \pm 2i$.

The eigenvalue $\lambda = 2i$ gives eigenvector $\mathbf{v} = (a,b)$ with

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (1-2i)a - 5b \\ a - (1+2i)b \end{bmatrix}$, so if we allow $b$ to be arbitrary, we obtain $a = (1+2i)b$ and solution $\mathbf{v} = ((1+2i)b, b) = (1+2i, 1)b$. Take $b$=1 and obtain complex solution

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{v}e^{2it} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix} (\cos 2t + i\sin 2t) = \begin{bmatrix} \cos 2t + i\sin 2t + 2i\cos 2t - 2\sin 2t \\ \cos 2t + i\sin 2t \end{bmatrix} =$

$= \begin{bmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + i \begin{bmatrix} \sin 2t + 2\cos 2t \\ \sin 2t \end{bmatrix}$, which gives two independent real solutions (take real and imaginary parts of this complex solution), so the general solution is

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t + 2\cos 2t \\ \sin 2t \end{bmatrix}$.