Quiz 1 **Math 221H** **Fall 2008**

Name:______________________________ **Score:**____________

Instructions: Show your work in the spaces provided below for full credit. You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Notes or text in *any* form are not allowed. The only electronic equipment allowed is a calculator.

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(7) **1.** (Exer. 3.1.30) (a) Show that $y_1 = x^3$ and $y_2 = |x^3|$ are linearly independent solutions on the real line of the equation $x^2y'' - 3xy' + 3y = 0$. (b) Verify that $W(y_1, y_2)$ is identically zero.
SOLUTION. (a) We have $x^2(x^3)'' - 3x(x^3)' + 3x^3 = 6x^2x - 3x \cdot 3x^2 + 3x^3 = 0$. So y_1 is a solution for all x and y_2 for $x \geq 0$, since then $y_1(x) = x^3 = y_2(x)$.

If $x < 0$, $|x^3| = -x^3$ and we check that $x^2(-x^3)'' - 3x(-x^3)' + 3(-x^3) = -6x^2x + 3x \cdot 3x^2 - 3x^3 = 0$, so y_2 is a solution for all x.
If x^3 were a multiple of $|x^3|$, then the coefficient would be 1 for $x > 0$, yet -1 if $x < 0$, since $|x^3| = -x^3$ for negative x. This is impossible, so these functions are linearly independent.
(b) For $x \geq 0$, $y_2(x) = x^3$, so $W(y_1, y_2) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 3x^5 - 3x^5 = 0$.
For $x < 0$, $y_2(x) = -x^3$, so $W(y_1, y_2) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = -3x^5 + 3x^5 = 0$.

(7) **2.** (Exer. 3.2.38) Use reduction of order to find a second linearly independent solution $y_2(x)$ of the DE $x^2y'' + xy' - 9y = 0$, $x > 0$, given $y_1(x) = x^3$.
SOLUTION. Assume that $y_2(x) = v(x)\, y_1(x) = vx^3$ and calculate

$$\begin{array}{rl} \{y_2 = vx^3\} & (-9) \\ \{y_2' = v \cdot 3x^2 + v'x^3\} & x \\ \{y_2'' = v''x^3 + 6v'x^2 + 6vx\} & x^2 \end{array}$$

Multiply both sides of the equations by the outside terms and add them up, setting the left-hand sides equal to zero and obtain

$$0 = v''x^5 + 6v'x^4 + 6vx^3 + v \cdot 3x^3 + v'x^4 - 9vx^3 = v''x^5 + 7v'x^4.$$

Cancel x^4 and substitute $u = v'$ to obtain $du/u = -7dx/x$, so that integrating and taking exponentials gives $u = -1/x^7$ and so $v = \int u\,dx = -x^{-6}/6$. Hence,
$y_2 = -x^{-6}x^3/6 = -x^{-3}/6$.
Or we could simply drop the constant coefficient and set $y_2 = x^{-3}$.

(6) **3.** (Exer. 3.3.13) Find the general solution to the DE $9y''' + 12y'' + 4y' = 0$.
SOLUTION. The characteristic equation is

$$9r^3 + 12r^2 + 4r = 0 = r\left(9r^2 + 12r + 4\right).$$

The roots of the second factor are $\frac{-12 \pm \sqrt{144 - 4 \cdot 9 \cdot 4}}{18} = -\frac{2}{3} \pm 0$. Thus the roots to the characteristic equation are $r = 0, -2/3, -2/3$ and the general solution is given by

$$y(x) = c_1e^{0 \cdot x} + c_2e^{-\frac{2}{3}x} + c_3xe^{-\frac{2}{3}x} = c_1 + (c_2 + xc_3)\, e^{-\frac{2}{3}x}.$$